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# Fractal Geography

**André Dauphiné**

**ISTE**

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## Table of Contents

|                                                                                       |    |
|---------------------------------------------------------------------------------------|----|
| <b>Introduction . . . . .</b>                                                         | xi |
| <b>Chapter 1. A Fractal World . . . . .</b>                                           | 1  |
| 1.1. Fractals pervade into geography . . . . .                                        | 2  |
| 1.1.1. From geosciences to physical geography . . . . .                               | 3  |
| 1.1.2. Urban geography: a big beneficiary . . . . .                                   | 6  |
| 1.2. Forms of fractal processes . . . . .                                             | 10 |
| 1.2.1. Some fractal forms that make use of the principle<br>of allometry . . . . .    | 11 |
| 1.2.2. Time series and processes are also fractal . . . . .                           | 12 |
| 1.2.3. Rank-size rules are generally fractal structures . . . . .                     | 14 |
| 1.3. First reflections on the link between power laws and fractals . . . . .          | 14 |
| 1.3.1. Brief introduction into power laws. . . . .                                    | 15 |
| 1.3.2. Some power laws recognized before the fractal era. . . . .                     | 17 |
| 1.4. Conclusion . . . . .                                                             | 19 |
| <b>Chapter 2. Auto-similar and Self-affine Fractals . . . . .</b>                     | 21 |
| 2.1. The rarity of auto-similar terrestrial forms . . . . .                           | 22 |
| 2.2. Yet more classes of self-affine fractal forms and processes . . . . .            | 24 |
| 2.2.1. Brownian, fractional Brownian and multi-fractional<br>Brownian motion. . . . . | 25 |
| 2.2.2. Lévy models. . . . .                                                           | 32 |
| 2.2.3. Four examples of generalizations for simulating<br>realistic forms. . . . .    | 35 |
| 2.3. Conclusion . . . . .                                                             | 37 |

|                                                                                                                                                            |           |
|------------------------------------------------------------------------------------------------------------------------------------------------------------|-----------|
| <b>Chapter 3. From the Fractal Dimension to Multifractal Spectrums . . . . .</b>                                                                           | <b>39</b> |
| 3.1. Two extensions of the fractal dimension:                                                                                                              |           |
| lacunarity and codimension . . . . .                                                                                                                       | 40        |
| 3.1.1. Some territorial textures differentiated by their lacunarity. . . . .                                                                               | 40        |
| 3.1.2. Codimension as a relative fractal dimension . . . . .                                                                                               | 41        |
| 3.2. Some corrections to the power laws: semifractals, parabolic<br>fractals and log-periodic distributions . . . . .                                      | 43        |
| 3.2.1. Semifractals and double or truncated Pareto distributions . . . . .                                                                                 | 43        |
| 3.2.2. The parabolic fractal model. . . . .                                                                                                                | 45        |
| 3.2.3. Log-periodic distributions. . . . .                                                                                                                 | 46        |
| 3.3. A routine technique in medical imaging: fractal scanning . . . . .                                                                                    | 48        |
| 3.4. Multifractals used to describe all the irregularities of a set<br>defined by measurement . . . . .                                                    | 50        |
| 3.4.1. Definition and characteristics of a multifractal . . . . .                                                                                          | 50        |
| 3.4.2. Two functions to interpret: generalized dimension spectrum<br>and singularity spectrum . . . . .                                                    | 52        |
| 3.4.3. An approach that is classical in geosciences but exceptional<br>in social sciences . . . . .                                                        | 54        |
| 3.4.4. Three potential generalizations . . . . .                                                                                                           | 56        |
| 3.5. Conclusion . . . . .                                                                                                                                  | 57        |
| <b>Chapter 4. Calculation and Interpretation of Fractal Dimensions . . . . .</b>                                                                           | <b>59</b> |
| 4.1. Test data representing three categories of fractals: black<br>and white maps, grayscale Landsat images and pluviometric<br>chronicle series . . . . . | 60        |
| 4.2. A first incontrovertible stage: determination of the fractal class<br>of the geographical phenomenon studied . . . . .                                | 62        |
| 4.2.1. Successive tests using Fourier or wavelet decompositions. . . . .                                                                                   | 63        |
| 4.2.2. Decadal rainfall in Barcelona and Beirut are fractional<br>Gaussian noise . . . . .                                                                 | 73        |
| 4.3. Some algorithms for the calculation of the fractal dimensions<br>of auto-similar objects . . . . .                                                    | 75        |
| 4.3.1. Box counting, information and area measurement<br>dimensions for auto-similar objects . . . . .                                                     | 75        |
| 4.3.2. A geographically inconclusive application from perception. . . . .                                                                                  | 78        |
| 4.4. The fractal dimensions of objects and self-affine processes . . . . .                                                                                 | 80        |
| 4.4.1. A multitude of algorithms . . . . .                                                                                                                 | 80        |
| 4.4.2. High irregularity of decadal rainfall for Barcelona<br>and Beirut . . . . .                                                                         | 84        |
| 4.5. Conclusion . . . . .                                                                                                                                  | 85        |

|                                                                                                                                  |     |
|----------------------------------------------------------------------------------------------------------------------------------|-----|
| <b>Chapter 5. The Fractal Dimensions of Rank-size Distributions.</b>                                                             | 87  |
| 5.1. Three test series: rainfall heights, urban hierarchies<br>and attendance figures for major French museums                   | 88  |
| 5.2. The equivalence of the Zipf, Pareto and Power laws                                                                          | 89  |
| 5.3. Three strategies for adjusting the rank-size distribution curve                                                             | 92  |
| 5.3.1. A visual approach using graphs                                                                                            | 92  |
| 5.3.2. Adjusting the only linear part of the curve                                                                               | 95  |
| 5.3.3. Choosing the best adjustment, and therefore the most<br>pertinent law                                                     | 96  |
| 5.3.4. Which rank-size distribution should be used for Italian<br>towns, the main French agglomerations and all French communes? | 98  |
| 5.4. Conclusion                                                                                                                  | 101 |
| <b>Chapter 6. Calculation and Interpretation<br/>of Multifractal Spectrums</b>                                                   | 103 |
| 6.1. Three data sets for testing multifractality: a chronicle series,<br>a rank-size distribution and satellite images           | 104 |
| 6.2. Distinguishing multifractal and monofractal phenomena                                                                       | 104 |
| 6.2.1. An initial imperfect visual test                                                                                          | 105 |
| 6.2.2. A second statistical test: generalized correlation dimensions                                                             | 107 |
| 6.3. Various algorithms for calculation of the singularity spectrum                                                              | 111 |
| 6.3.1. Generalized box-counting and variogram methods                                                                            | 111 |
| 6.3.2. Methods derived from wavelet treatment                                                                                    | 112 |
| 6.3.3. Interpretation of singularity spectrums                                                                                   | 113 |
| 6.4. Possible generalizations of the multifractal approach                                                                       | 116 |
| 6.5. Conclusion                                                                                                                  | 118 |
| <b>Chapter 7. Geographical Explanation of Fractal Forms<br/>and Dynamics.</b>                                                    | 121 |
| 7.1. Turbulence generates fractal perturbations and multifractal<br>pluviometric fields                                          | 122 |
| 7.2. The fractality of natural hazards and catastrophic impacts                                                                  | 126 |
| 7.3. Other explanations from fields of physical geography                                                                        | 128 |
| 7.4. A new geography of populations                                                                                              | 129 |
| 7.5. Harmonization of town growth distributions                                                                                  | 131 |
| 7.6. Development and urban hierarchies                                                                                           | 132 |
| 7.7. Understanding the formation of communication<br>and social networks                                                         | 136 |
| 7.8. Conclusion                                                                                                                  | 137 |

|                                                                                                             |            |
|-------------------------------------------------------------------------------------------------------------|------------|
| <b>Chapter 8. Using Complexity Theory to Explain a Fractal World . . . . .</b>                              | <b>139</b> |
| 8.1. A bottomless pit debate. . . . .                                                                       | 140        |
| 8.2. General mechanisms for explaining power laws . . . . .                                                 | 143        |
| 8.3. Four theories on fractal universality . . . . .                                                        | 144        |
| 8.3.1. Critical self-organization theory . . . . .                                                          | 144        |
| 8.3.2. Béjan's constructal theory. . . . .                                                                  | 151        |
| 8.3.3. Nottale's scale relativity theory . . . . .                                                          | 153        |
| 8.3.4. A general theory of morphogenesis . . . . .                                                          | 154        |
| 8.3.5. Chaos and fractal analysis theory . . . . .                                                          | 163        |
| 8.4. Conclusion . . . . .                                                                                   | 164        |
| <b>Chapter 9. Land-use Planning and Managing a Fractal Environment . . . . .</b>                            | <b>167</b> |
| 9.1. Fractals, extreme values and risk . . . . .                                                            | 168        |
| 9.1.1. Under-estimated hazards in preliminary risk assessments . . . . .                                    | 168        |
| 9.1.2. Fractal networks, fighting epidemics and Internet breakdowns . . . . .                               | 171        |
| 9.2. Fractals, segmentation and identification of objects in image processing. . . . .                      | 173        |
| 9.2.1. New image processing tools . . . . .                                                                 | 173        |
| 9.2.2. Some little-used fractal approaches using a GIS. . . . .                                             | 177        |
| 9.3. Fractals, optimization and land management . . . . .                                                   | 177        |
| 9.4. Fractal beauty and landscaping . . . . .                                                               | 179        |
| 9.5. Conclusion . . . . .                                                                                   | 180        |
| <b>Conclusion . . . . .</b>                                                                                 | <b>183</b> |
| C.1. Some tools and methods for quantifying and qualifying multiscale coarseness and irregularity . . . . . | 184        |
| C.2. A recap on geographical irregularities and disparities . . . . .                                       | 186        |
| C.3. A paradigm that gives rise to new land-use management practices . . . . .                              | 189        |
| <b>Appendices . . . . .</b>                                                                                 | <b>191</b> |
| A.1. Preliminary thoughts on fractal analysis software . . . . .                                            | 191        |
| A.2. Instructions for the following programs. . . . .                                                       | 192        |
| A.3. Software programs for the visual approach of a satellite or cartographic series or image . . . . .     | 193        |
| A.4. Software programs for calculating fractal dimensions for a chronicle or frequency series . . . . .     | 198        |
| A.5. Software programs for calculating the fractal dimensions of a satellite image or map. . . . .          | 208        |

|                                                                                                     |     |
|-----------------------------------------------------------------------------------------------------|-----|
| A.6. Software programs for calculating multifractal spectrums<br>of a series and an image . . . . . | 213 |
| <b>Bibliography</b> . . . . .                                                                       | 221 |
| <b>Index</b> . . . . .                                                                              | 239 |



## Introduction

“To the scientific mind, all knowledge is an answer to a question. Without questions there cannot be any scientific knowledge. Nothing is obvious. Nothing is given. Everything is constructed.”

Gaston Bachelard, 1938

Geographers deal with three types of issues.

First, they investigate the relative locations of populations, their activities such as industries or services, or their cultural properties. In order to respond to these topics, they make use of a range of theories on localization, which have been principally elaborated by spatial economists. For example, the theory developed by J.H. Von Thünen opened the way for cultural organization to be expressed as rings around a market before restructuring urban economies. A. Weber and W. Alonso concentrated on industrial localization, while W. Christaller and A. Lösch focused on localization of services in key areas. All of these theories are presented and applied in the combined works of M. Fujita *et al.* [FUJ 99] and P.-Ph. Combes *et al.* [COM 06]. Nevertheless, they remain incomplete, as they do not take account of all physical and cultural factors.

Second, geographers study the relationships between human societies and their physical surrounding, especially in classical geography. In the past, the emphasis was on the more or less determinant role of natural environments; now social and cultural dimensions have begun to take priority. In these studies, fewer theories are expounded and they are also less well developed. Geographers thus make use of

their preferred empirical research in underlying theoretical contexts, for example Lamarck's or Darwin's theories of evolution. This is well proven by reading any of the major theses on climatic geomorphology or rural geography. The principle of gradualism has for a long time reigned unchallenged, with all reliefs and rural landscapes evolving slowly and uniformly.

Finally, geographers focus on terrestrial forms, whether they are physical, biological or socioeconomic. Seen from a particular point of view, geography is thus a morphology. It becomes morphogenetic when its followers rely on the emergence and evolution of terrestrial forms, such as landscape, town, region and continental transformations. In order to account for the emergence and succession of these forms, theories from physical science are used, such as Turing's theory, which we will make use of in Chapter 8. Life sciences take these theories and improve on them, as illustrated by the works of D'Arcy Thomson [DAR 94] and V. Fleury [FLE 09]. They are less common in social sciences, despite the recent success of the Schelling model [SCH 80], which renews our understanding of sociospatial segregation.

Whatever the question, and therefore whatever the geographical project used – i.e. whether classical or contemporary – spatial discontinuities, differentiations and disparities are always at the heart of the geographical project, as proclaimed by R. Brunet [BRU 68]. Geographers have always been united in considering fronts and frontiers, irregularities, diversities and disparities of territories as being central to their scientific programs. Certainly, every geographer follows his or her own methodology or favors one technique over another, but all concentrate their thoughts on the issues that they judge to be the most important.

Furthermore, these irregularities and spatial disparities are observed on all scales. P. Claval [CLA 68] strongly emphasized this in a reference work; but he was not alone. All geographers are sensitive to the multiscale characteristics of the behaviors being studied. The drafting of any job application for any university confirms this. In France, this theme is even followed in the "Aggregation" program [BAU 04]. The majority of geographical studies reflect on disparities and scales. Two recent contributions bring this to mind: the work of E. Sheppard [SHE 04] and the theme of the *Géopoint* 2010 symposium. Many others can also be quoted from all domains of geography.

This interest in multiscale systems is, moreover, not particular to geography, but is found in a great many disciplines. Physicists have been plowing this field since Boltzman, who connected Clausius entropy, a macroscopic width, to the configuration of microstates. More generally, 20<sup>th</sup> Century statistical physics, which accords a priority to microstates, has been constructed in order to account for the laws of macrophysics discovered in the 19<sup>th</sup> Century. The contemporary work of

M. Laguës and A. Lesne [LAG 03], which is dedicated to this discipline, thus offers a host of recommendations to geographers and social science specialists.

It is the same for other disciplines. Some similar contributions come from the works of economists and sociologists faced with the double problem of combining inequality with scale. Long debates still divide the advocates of micro- versus macroeconomics or question the role of methodological individualism in sociology. Furthermore, in the area of life sciences, the principle of natural selection has changed. From the Darwinian level of species it is now identified as active at the cellular level in post-Darwinian theory.

Yet despite this, the fractal paradigm, which is the subject of this small collection of works, provides a new view of the disparities and at the same time of multiscale phenomena. It constitutes a sort of hinge, or bridge, which links these two main questions. It opens the way to a combined analysis of irregularity and level. From this fact all studies into morphology and territorial morphogenesis and, more generally, geographic studies are updated.

Initially, B. Mandelbrot [MAN 75] describes irregular mathematical objects as fractals, whose irregularities are the same on all scales. These abstract objects are said to be auto-similar and scale invariant. These two very similar concepts are not, however, synonyms, which is why they merit some preliminary discussion in this introduction.

The first concept, auto-similarity or internal similitude, is geometric in nature. It refers to an object comprised of sections that are copies of the object itself, which signifies that the whole is identical to its parts. Each section can also be broken down into subsections identical to itself. In reality, this iterative process, which is repeated infinitely in fractal mathematics, always has an upper and lower limit. In order to designate these double-limit fractals as “real” or “known”, B. Mandelbrot preferred the term pre-fractal, but this concept was abandoned. These two limits however, should not be confused with resolution and range, which are two other limits that are dependent on observation and not the nature of the object.

Directly linked to this, the concept of scale invariance, or invariance by dilatation, is more statistical in nature. It indicates that a similar characteristic is observed at all scales. The fractal dimension is a measure of the rate of variation of data from one level to another, which is why B. Sapoval [SAP 97] considers fractals to be a geometry of probabilities. Thus, contrary to what some users believe, the fractal approach is not reduced to a geometric approach (the study of shapes).

Furthermore, scale invariance is a more general concept than auto-similarity. Scale invariance certainly encompasses auto-similarity, but also long-term or long-

range dependence. Identification of this dependence is a result of attributing fractal characteristics to analyzed observations.

There is an abundance of literature on fractals that shows a number of authors prefer to think in terms of auto-similarity, geometry or the Koch or Peano shapes, while financial specialists, who are passionate about share prices, almost exclusively reflect on scale invariance, without recourse to geometry. In this work we try not to favor one viewpoint over the other, since we share B. Sapoval's idea, which in a way boils down to reuniting space and time.

This inherent link between geometry and probability also indicates that all scales are significant for the system under consideration. Better still, for a proper understanding of a fractal phenomenon, physicists have demonstrated that the interactions between scales have greater determinance than those between elements. This is doubtless true for other biological and social phenomena.

Before we consider moving away from mathematics for the first three chapters, remember that a fractal has a higher Hausdorff-Besicovitch dimension than topological dimension. A straight or broken line has a topological dimension equal to 1. However, if it is irregular, then its fractal dimension is greater than 1. Similarly, an irregular area will have a higher fractal than topological dimension of 2.

The Hausdorff-Besicovitch dimension, which measures the extent of this irregularity, is the logarithmic ratio between the number of internal homotheties of the object  $N$  and the inverse of this ratio,  $r$  [LOP 10]. This definition is sufficient to convey the fact that this dimension is difficult to determine for concrete physical objects. In order to overcome this problem, scientists from different disciplines use numerous algorithms. This explains the richness and complexity of Chapters 4, 5 and 6, which, without intending to be exhaustive, outline the vast number of fractal and multifractal dimensions.

The scope of application of fractals is widening beyond mathematics. First, they are described as physical or living shapes, such as a coastal outline, a river system or as a neural maze. At the same time, researchers are moving away from the study of shapes to that of processes in all branches of science, whether physical or economical, such as in linguistics. The fractal treatment of chronological sequences has been generalized, most notably in the areas of climatology and financial economics.

Several techniques have been developed in order to calculate the fractal dimension, depending on the circumstances encountered. Nevertheless, a fractal object always has a higher Hausdorff-Besicovitch dimension than its topological dimension. Too many studies seem to ignore this rule and instead consider the

fractal dimension from the slope of a log-log or bilogarithmic adjustment; hence various authors produce examples where fractal dimensions are less than the topological dimensions of the objects being studied. Without exception, as we will see later in this book, the fractal dimension is calculated from the slope of the log-log graph, but it is not equal to this slope. It is derived from this by using a formula that is adapted to each fractal category.

Scientists are no longer content to qualify an object, to simply describe its irregularity in terms of an overall fractal dimension, equivalent to a statistical mean. They develop theories in order to explain fractality in global terms. Over recent decades, fractality has presented itself as a veritable paradigm, with its techniques, methods and theories.

In French geography, the pioneering works were those of A. Dauphiné [DAU 90-91, DAU 95] and P. Frankhauser [FRA 91, FRA 94]. Based on their work, this technique was expanded upon by the southern teams at the UMR ESPACE and the UMR Théma in Besançon. Other theoretical geography research teams have also richly contributed to this area, notably in Caen, Rouen, Grenoble, Pau, Strasbourg and Paris; not forgetting some brilliant “retirees” [BRI 04] and some French-speaking colleagues in Milan, Louvain, Switzerland and Québec. Today, over 100 French and French-speaking geographers understand and occasionally practice the fractal approach. This compilation work owes much to their research.

Quite clearly, the fractal approach is not exclusive to French geography. Some works were also expanded upon in geography in the English-speaking world following the publication edited by N. Lam and I. De Cola [LAM 93]. This has already uncovered explanations on fractal hydrological networks or on the localization of key places. The authors then called upon rigid explanations to account for irregularities measured using the fractal dimension. Additionally, in this landmark publication, two interventions demonstrate the benefit that geographers are able to draw from multifractal formalism. In particular, these authors provided geographers with computer programs, written in FORTRAN, which enabled fractal dimensions to be calculated and the first simulations of terrestrial reliefs to be created.

This publication was put together following seminal articles on ecology by P.A. Burrough [BUR 81], urban geography by M. Batty [BAT 85], geomorphology by M.F. Goodchild [GOO 87], and many others. It is difficult to name them all without forgetting some. Since these initial works, other schools have joined this vast movement, notably Chinese geographers from Peking University, who are very active in this field of research.

Well beyond geography, the fractal paradigm has flooded into all disciplines – the physical and chemical sciences [GOU 92], the life sciences, engineering

[ABR 02] and the economic and economic sciences [MAN 97, LEV 02]. Even philosophers are examining this paradigm, as attested by A. Boutot's book [BOU 93]. In the final chapters of his book, this philosopher advances a very informative classification for fractals.

Indeed, for the individual social sciences, the science and social science indices lists more than 20,000 works published in 2010, 10,574 in 2011 amongst a total of 340,201. We have not read all of these works, but they do provide valuable assistance to the geographer, since they introduce new methods and techniques. Furthermore, the most recent of these advance some of the formal and general theories that bypass rigid explanations and are too reductionist.

As with all compilations, this work is provisional. It is written for use by geographers and researchers from similar disciplines, such as ecologists, economists, historians and sociologists, which is why we are adopting a classical format, tackling descriptions of fractal phenomena before explaining them in later chapters. This description relies on a series of observations. However, there is nothing to prevent these exercises from being carried out on results from simulation, such as images produced by multiagent systems or macromodels that couple differential equations.

The first part of this book sets out a fractal panorama avoiding mathematical formalization, which is the unifying theme of the second part, as much as possible. Having observed the diversity and ubiquity of a fractal geographical world in Chapter 1, in Chapter 2 we will distinguish auto-similar fractals from those that are self-affine. Chapter 3 concludes this first part, with a rapid tour of the tools that have come about from the fractal dimension, from lacunarity to multifractal spectrums.

The next three chapters present a number of algorithms that are not the most widely used, but are best matched to the phenomena studied within social sciences and geography. After a study of auto-similar and self-affine fractals in Chapter 4, Chapter 5 deals with the fractal dimension of the rank-size rules. Generations of geographers, notably urban geography specialists, have relied on these rank-size rules. They are still being hotly debated and so merit special attention. Finally, multifractal formalism is tackled in Chapter 6. In all of these chapters, case studies enable us to judge the relevance of algorithms applied to "concrete" situations, with sets of data illustrating some diverse phenomena in one or two dimensions.

Above and beyond description, all geographers wish to bring about an explanation. The third part of this book comprises two chapters focused on comprehension, first disciplinary and second more general, on fractal shapes and processes. In Chapter 7 we show how it is possible to interpret fractal dimensions and power laws in geographical and, more generally, disciplinary terms. This chapter already demonstrates that the fractal is not simply a descriptive tool. It

engages the geographer in new, illustrative ways. Pursuing our efforts at abstraction; in Chapter 8 we shed light on a number of general formal theories which bypass disciplinary frontiers. These controversial complexity theories will, despite their ambiguity, direct us towards a world where fractality rules.

Finally, the last chapter shows that the fractal approach raises questions about and provides answers regarding environmental management, or the tackling of problems related to town planning. Fractal reasoning is not a simple game played out on a computer screen or a highly theoretical exercise, but instead opens up some very real problems. Managing a fractal world imposes changes. This chapter is completed with a long appendix presenting computer programs written in the *Mathematica* language. Not yet optimized, these programs offer the potential to be used directly by young geographers who are interested in implementing these approaches.

The roots of this work are long-standing. It has thus benefited from constructive comments from our young southern colleagues, exchanges during thesis presentations and in meetings of the Dupont group, notably in the presence of the Avignon geographers, and from the *Théo Quant* symposiums. We would like to thank all of these contributors for their assistance, particularly Eric Bailly and Damienne Provitolo, who checked our occasionally too condensed reasoning and enabled us to improve the comprehension of this text, while patiently requesting clarifications. Whatever remains vague is quite clearly down to us.



## Chapter 1

# A Fractal World

“Fractals are mathematical objects, whether naturally or man made,  
which can be described as irregular, coarse, porous or fragmented,  
and which, furthermore, possess these properties  
to the same extent on all scales.”

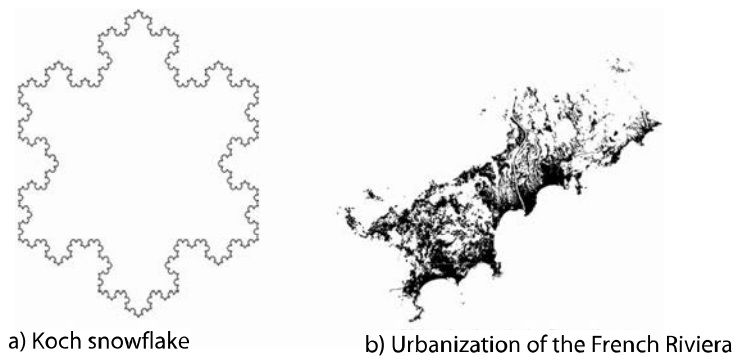
Benoît Mandelbrot, 1975

The diversity, coarseness and irregularity of land are all at the heart of geographical projects. It is therefore not surprising that the fractal approach enriches this discipline, since the fractal dimension was originally a measure of an object's irregularity and shape, as emphasized by the definition provided by B. Mandelbrot [MAN 75]. The calculation of a fractal dimension therefore enables the irregularity and, in a way, the complexity of all land to be defined.

Following a rapid and therefore mostly incomplete French-language census of the fractal approach in geography in section 1.1, in section 1.2 we show how this approach, which is initially focused on shape, is extended to mechanisms, processes and functional structures that are realized through a series of frequential data or a rank-size rule. The links between fractals and power law are detailed in section 1.3. Without aiming to be completely historical, this chapter highlights the pioneering works of the French-speaking geographers who set the fractal approach in motion. More recent research is referenced in later chapters.

## 2 Fractal Geography

Before tackling these issues, we recall that a fractal object possesses a number of properties. It is certainly irregular and fragmented, as emphasized in all known definitions, but is also either auto-similar or self-affine. An object that is irregular without being auto-similar or self-affine is not a fractal. An auto-similar phenomenon has perfectly identical sections throughout. It is often constructed through iteration of a basic pattern, as with the Koch snowflake (see Figure 1.1). Occasionally, the initial pattern is deformed in one or more directions. The phenomenon is then no longer auto-similar, but self-affine. The urbanization of the French Riviera (see Figure 1.1), which is spread along axes that are parallel and perpendicular to the coast line, is an example of this.



**Figure 1.1.** *Two fractal forms*

Additionally, fractals go hand-in-hand with power laws, since these are the only scale-invariant laws. Finally, the fractal dimension is always greater than the topological dimension. In practice, this fractal dimension is calculated from the value of the power law slope; and yet this slope, which is also known as a scaling parameter, should not be confused with a fractal dimension.

### 1.1. Fractals pervade into geography

In his early works, in order to define a fractal, B. Mandelbrot made reference to familiar geographical objects. In his widely-available publication *Les Objets Fractals: Forme, Hasard et Dimension* (Fractals: Forms, Chance and Dimension) [MAN 75], he visited or revisited some geographical problems, such as the length of the Brittany coast lines, which were often tackled in earlier articles, particularly by L.F. Richardson [RIC 61]. Then, in *The Fractal Geometry of Nature* [MAN 77], he devoted whole chapters to hydrographical networks and relief forms.

His contribution was not limited to the study of spatial morphology; he was also concerned with evolution and rhythm, and therefore temporal laws and morphogenesis. In a short time he applied fractal formalism to time series, which were representative of process, by publishing some innovative works on stock market trends, such as the evolution of the price of cotton.

Following these pioneering works, it is now accepted that most terrestrial forms, as well as their origins, are fractal. No geographical phenomena are excluded from this approach. In the examples that follow, we provide academic and traditional distinctions, both in physical and human geography, although the fractal approach abolishes this distinction and reunites these two categories of objects.

#### **1.1.1. *From geosciences to physical geography***

In physical geography this reasoning, which has become the norm in the geosciences [DUB 95, KRU 94, SHO 89], is now being used in climatology, hydrology, biogeography and geomorphology research.

##### **1.1.1.1. *The omnipresence of fractals in meteorology – climatology and hydrology***

Geography is not insular or closed. Geographers communicate with researchers from similar disciplines, who often show the way forward. For example, since too few geographers and climatologists had taken ownership of the fractal paradigm [DAU 98], in a short time whole meteorological colleges were devoting all their efforts to writing articles and theses [BIA 04, HAL 01, HUB 89, LAD 93, LOV 82, SCH 87, TES 93]. Thus we had the opportunity to serve on the thesis jury of P.H. Ladoy, who drew our attention to the fractality of climatological variables.

Since his first works, S. Lovejoy [LOV 82] has demonstrated that the areas and perimeters of cloud formations are fractal. Since then, meteorologists have extended this fractal formalism to other climatic descriptors, but also to pluviometric and thermal rhythms, since the scale dependence on meteorological structures is permanent. The scale ratio can easily reach  $10^9$ , from the size of a drop of water to that of Hadley cells.

More recently, meteorologists have been making good use of multifractal formalism in order to understand the organization of temperature and pressure fields. They have emphasized that fractal formalism applies to simple bodies, but that the meteorologist nearly always associates a measurement with these bodies. Temperature, pressure and rain fields bring intensity to a body of weather stations. It is thus useful to apply multifractal formalism to these fields, as with population density charts or commercial areas. We will develop this area in Chapters 3 and 6.

#### 4 Fractal Geography

Additionally, meteorologists are at the heart of the interest in relative codimension, which we define in Chapter 3.

Despite everything, it was in geohydrology that fractals were perceived to be most relevant, notably in France with the works of D. Delahaye [DEL 02] and P.H. Martin [MAR 04]. Without a doubt, we need to recognize the significance of previous research carried out by R.E. Horton [HOR 45] and A.N. Stralher [STR 45] on the charting of river systems. This opened the way to a fractal vision of this discipline, as well as the need to promote effective flood management, taking into account not just a simple river section, but the whole hydrographic network of a river basin. This operational necessity has forced recourse to multiscale practices, which rely on fractal formalism.

Very soon, hydrologists established a power law between the length of river networks and their river basin area. Hack's law links the length of the principal drain to the surface of the basin using a power law. Recently, some algorithms, which have been incorporated into GIS (Geographical Information Systems) geographical systems, have calculated the fractal dimensions of river networks based on digital terrain modeling. Thus J. Douvinet *et al.* [DOU 08] have shown that the relationship between the maximum surface flow and the basin surface follows a power law where the scaling parameter is equal to 0.54. They have thus proposed a new set of indices, which show the effect of basin size in brackets in order to compare the efficiency of river networks. Furthermore, these hydrological geographers are not confined to their own discipline, but invest in many others, as many of them are also talented karstologists.

##### 1.1.1.2. *Rarer pioneering works from geomorphology and ecology*

Notwithstanding recent studies, which have occasionally been conducted by atypical researchers, French geomorphology has long remained on the fringes of this vast current. Exceptions to this are H. Regnaud [REG 93] and the Caen geomorphologists [HAU 02, DEL 02], who as a result of the fractal approach discovered a new set of fault directions north-west of the Paris Basin. These faults angled 90-100 north have complicated the traditional model of a tectonic evolution, which traditionally attached significance to single faults in the Armorican and Variscan directions.

This indifference to geomorphology is all the more surprising as geomorphologists were among the first to appreciate the role of fractals. The auto-similar characteristic of relief forms was implicitly accepted by all geologists who would use a hammer to visualize the scale of phenomena recreated in photographic documents produced by them.

An article by M.F. Goodchild and D.M. Mark [GOO 87] discussed the suitability of the fractal approach in geomorphology, not hesitating to criticize Mandelbrot's rather "hasty" generalizations. While recognizing the interest in descriptions of relief irregularity using the fractal dimension, these authors detailed the limitations of the fractional Brownian model, which we will cover in depth in Chapter 2, in understanding the organization of natural terrestrial forms.

Furthermore, the number of related fractal studies into earthquakes can be counted in the hundreds. The Gutenberg-Richter law on earthquake magnitude and the Omori-Utsu law, which measures aftershocks after an earthquake, are both power laws. The same applies to spatial fault distributions and the epicenters of earth tremors [DUB 95].

Geologists are not content with descriptive research. They have linked the fractality of the seismicity of various physical models, particularly those relating to percolation [ALL 82] and ultimately critical self-organization theory. They have proved that seismicity and earth tremors are characterized by the absence of a dominant scale. All temporal and spatial scales contribute equally to global seismicity.

Fractal simulation was first applied to relief forms. There are computer programs available in the pioneering works edited by N.S.-N. Lam and L. De Cola [LAM 93]. Also, the beautiful relief image simulations, which were used to illustrate works published in the 1990s, have undeniably contributed to the success of fractals, bearing as they did a striking resemblance to the true relief forms that geomorphologists explored during their field visits.

Finally, and still on the subject of geomorphology, the Canadian school very quickly became interested in the fractal paradigm. It continues to provide quality output, as testified by A. Beaulieu's thesis on fractal anisotropy [BEA 04], which followed in the footsteps of A. Robert and A.G. Roy [ROB 93] and the geologist H. Gaonac'h [GAO 92]. This examined the fractal characteristics of volcanic lava casts from samples taken from the peak of the furnace and from Etna. However, it was not able to provide a satisfactory explanation.

In biogeography, the works of geographers are even less commonplace, despite the early studies by P.A. Burrough [BUR 83a] on soil distribution and D.M. Mark [MAR 84] on the branched organization of coral. They sometimes compared the advantages of the fractal approach with the geostatistical treatment.

Some ecologists and biologists do, however, diffuse the fractal approach in their disciplines [BAS 98, BUR 99, CHA 99]. They generally calculate fractal dimensions using the variogram method, since they have control over the formalism of

geostatistics. We also owe a debt of thanks to marine biologist L. Seuront [SEU 10], for producing an excellent manual on fractal and multifractal formalism, which was influential when writing Chapter 2 of this book.

In addition, in France, it is worth mentioning the originality of the works of the “Montpellier school” in the field of fractal simulation of vegetation. They have simulated the seasonal and year-on-year evolution of vegetation landscapes, demonstrated that the fractal approach sometimes bypasses the area of fundamental research, and have offered practical methods in environmental management (Chapter 9).

### **1.1.2. Urban geography: a big beneficiary**

Within human geography, the fractal paradigm has principally conquered the field of urban studies and the area of networks. The fractal methodology does however, remain applicable to other fields of geographical knowledge.

#### *1.1.2.1. The fractality of perimeters and urban tissue*

Fractal dimension calculation is central to innovative research into urban forms. After numerous articles, M. Batty and P.A. Longley [BAT 94] devoted a substantial publication to this. They were not simply content to describe urban forms, but compared their fractal dimensions with those of forms obtained through simulation of the physical process of mass increase, such as the model of aggregation limited by diffusion (see Chapter 8).

In France, this trend was initiated by P. Frankhauser [FRA 94] and disseminated by his students. Numerous publications have highlighted the diversity of the fractal dimensions of urban areas (observed in black and white) that compare different stages of development. The geographers from the Besançon school have continued to plow the furrow by creating multiple algorithms that are then integrated into successive versions of the *Fractalyse* software.

The most original research has concerned the structures of urban districts. Such works have made use of fractal “scan” logic, which we will explain in Chapter 2 and which was made popular in image processing. This approach, which was applied by H.I. Qiu [QIU 99], has been spread in France by various authors who have been working alongside the school in Besançon.

This reasoning has been enhanced by the works of D. Badariotti [BAD 05] on the city of Strasbourg. He even developed some new indicators, such as the synthetic roughness or fractal coverage indices, which enabled him to establish some fruitful comparisons and to clearly distinguish between types of urban tissue. Also,

in research on the Milan urban area, M. Caglioni and G. Rabino [CAG 03] produced analyses on global tissue and agglomeration perimeters, before comparing the structures of the central districts of Milan-Bovisa and “Barona” using fractal dimensions.

Outside France, a number of urban geography specialists have compiled a list of population densities and other variables using digital terrain modeling. The fractal dimension has enabled them to define the spatial irregularity of cartographic phenomena. For example, D.W.S. Wong *et al.* [WON 99] analyzed social segregation in American towns using a fractal approach. It is a pity that these research developments have been little utilized by social geography specialists, since fractal instruments are highly efficient at measuring irregularities and disparities and are better for understanding the intensity of sociospatial segregation.

In Nice, E. Bailly [BAI 99] was applying the principles of fractal dimension in another area, simulating the growth of a town using a fractal over a base map. His model, which simulated building distribution, was a cellular automaton sketch that was engaged in a way hardly explored by M. Batty [BAT 94]. Cellular automaton or multi-agent system simulations of urban growth have been enriched as a result of increasingly complex case studies. The analysis of the Tokyo urban area produced by I. Matsuba and M. Namatame [MAT 03] is an example of this. A cost function was superimposed over an aggregation model limited by diffusion in order to simulate realistic urban growth. This is a good example of how a physical model can work alongside economic law.

Away from building studies, urban transport networks have given rise to fractal dimension calculations. L. Chapelon [CHA 96], N. François [FRA 97] and C. Genre-Grandpierre [GEN 99] have conducted numerous research projects on this theme. Since it had already been applied to the Paris Metro by L. Benguigui [BEN 95], this methodology was greatly appreciated by British researchers.

We should also mention studies into urban development, which are detailed in Chapter 9, and especially the analyses of pollution on the French Mediterranean road network by G. Maignant [MAI 02], which showed that the fractal dimensions of the web of urban routes are virtually equal to 1.20 Df for the towns of Nice, Marseille and Lyon.

Conversely, fractal studies into urban networks are less common [ARL 85, BAT 92, DAU 03c], although S. Arlinghaus showed very early on that the networks from central place theory can be faithfully reproduced through the simulation of a Koch fractal. The theory can even determine the fractal dimension, or Df, of the three main types of central place hierarchies recognized by W. Christaller, which correspond to market (Df = 1.26), transport (Df = 1.58) and administrative

( $D_f = 1.12$ ) principles. The administrative principle, which has minimum fractal dimension, therefore creates more regular networks, while the transport principle is a most highly irregular source of urban meshing.

Beyond this study, however, it is true that the approaches in terms of the Auerbach-Zipf or Pareto distributions of urban hierarchies (which have often been visited since the end of the 19<sup>th</sup> Century) have stemmed in part from fractality, even if the term *fractal* was not in use at the time. There is an explanation for this precedence. All fractal laws have a power law, and the rank-size rule is a power law.

Indeed, numerous power laws were discovered well before the term *fractal* had been suggested. This comment is further discussed in section 1.3 of this chapter and then in Chapter 5, which is completely devoted to rank-size rules.

Many comparative studies, whether in space [MOR 93] or time [GUE 93], have made use of the rank-size rule, which is the typical example of a power law. For a number of years, Pekin University geographers were famous for their numerous contributions [CHE 09, GAN 06, SON 02]. For example, they revisited the urban-rural thematic and made continual reference to multifractal formalism in order to understand the rank-size rule.

#### 1.1.2.2. *A recent application: the fractality of communication and social networks*

While urban studies were the focus of research into human geography, which made use of the fractal approach, other themes were also tackled by these tools. This is particularly apparent in the case of transport and communication networks, with their plethora of analyses into traffic and internet networks, and even social networks – even if the majority of these studies have been conducted by non-geographers [ABR 04, PAS 04].

Other networks, such as river or irrigation networks, have also been described using the fractal approach [BRI 04, THI 91]. Scale invariance is still being reported in the case of the world's shipping lanes [HU 09] and, on a national scale, M. Guida and F. Maria [GUI 07] have demonstrated that the Italian airport system is scale invariant; each airport is characterized by a rating that corresponds to the number of links it has with other airports. These ratings are ranked according to what is known as the double Pareto distribution, which is in fact equivalent to a semi-fractal, which is a model that we will define more precisely in Chapter 3. This same distribution is observed for Chinese and American airports and, more generally, for numerous transport networks.

In all network studies, fractal dimension calculations have initially been applied to the topology, particularly to the nodes, or peaks, of the networks. Three indicators are generally attached to each node. The first indicator, the degree, is simply the

number of links between one node and all the others. The second indicator, the centrality of intermediarity of a peak, or centrality, is defined as the total number of shorter connections that pass through this peak, from the very short connections between all pairs of peaks in the network [PAS 04]. Finally, the degree of peak *clustering* is defined as the ratio between the number of links between neighboring peaks and the number of potential links. For many networks, the values of these three parameters for each peak follows a power law, which is synonymous with fractality when several levels are recognized.

However, the fractal approach also applies to flows circulating in these networks. This is true of internet traffic, but more so of motor traffic flows in large agglomerations. Thus, X. Li and P. Shang [LI 07] classified traffic flows along the highway from Yuquan to Beijing by making use of multifractal formalism. With regards to internet traffic, P. Abry [ABR 04] distinguished between additive random walks, which are the cornerstone of monofractal behavior, and multifractal processes, which are multiplicative. He also affirmed that scale invariance is initially referenced over very short timescales of under a second, but also over longer periods of several seconds. These two scale-invariance phenomena can take root in anything between 10 microseconds and several hours, i.e. over seven decades, or levels.

#### 1.1.2.3. *New territories to visit: the fractality of population, rural landscapes... and even cultures*

While most fractal studies have concerned towns and networks, this formalism has been able to generate new knowledge, whatever the geographical theme envisaged. Thus, still in France, research concentrating on population has been encouraged, mainly by H. Le Bras [LEB 00]. The thesis written by M. Forriez [FOR 10a] is also available to geographers today. M. Forriez studied the fractality of populations counted in various countries, i.e. on a global scale.

This fractal formalism has been no less effective in studies examining migratory flows. For example, Zipf's law was derived from migratory flows in the United States, which affected more than 46 million people between 1995 and 2000 [LEV 10]. Equally relevant are researches conducted into mobility from exchanges using cell phones, which have virtually all inferred power law distributions on human movements from the localization of messages that have been sent and received. As with network analysis, it has been possible to use fractality equally well to characterize structures as well as flows.

Finally, fractality has been observed in numerous rural areas, as shown by the farmland study undertaken by B. Roland and C. Fleurant [ROL 04] and that of K. Berthier [BER 03] on the local scale of landscape. Fractality does not stop at

town gates, although a sudden change in the fractal dimension is a good indication of a boundary between town and country [KEE 03].

## 1.2. Forms of fractal processes

We can draw three conclusions from this brief and incomplete view on the works of essentially French-speaking geographers.

First, no rigorous and precise definition of a fractal exists. As with life, the definition of which remains the subject of debate, a fractal is instead recognizable from various characteristics. Auto-similarity, and by extension self-affinity – which we noted in the introduction as being complimentary to but distinguishable from scale invariance – is a first sign of fractality. These two characteristics must be observed on several levels, although how many are required has not been universally agreed. It would appear, however, that three levels constitutes a minimum number. Additionally, the form iteratively obtained must be irregular and fragmented. The third fractal property is that the Hausdorff-Besicovitch fractal dimension must be greater than the topological dimension, apart from in the case of some fractals, such as the “devil’s staircase”. Furthermore, since it is difficult to directly calculate the Hausdorff-Besicovitch dimension, fractalists of all persuasions have developed multiple algorithms in order to estimate the fractal dimension. These techniques, which are explained in Chapters 3, 4 and 5, lead to different results that may occasionally mislead a novice.

Second, this overview shows that Euclidean forms are singular cases in a universe of fractal forms. Certainly, Euclidean forms are not completely absent on a global or universal level, but they are less commonplace than fractal forms, which is not surprising. Indeed, once we accept that these forms may have any dimension in the range 0 to 4, it is easy to understand that Euclidean forms, whose dimensions are limited to whole numbers 0 (point), 1 (line), 2 (surface) and 3 (volume), are only islands in a vast dimensional ocean with the potential to be fractional. This is all the more true when certain fractal forms occasionally have a whole-number dimension value.

Third, fractal formalism is relevant not only to the study of terrestrial forms, but also to their dynamics and to studies into frequency, and therefore structure. This conclusion merits further investigation.

### 1.2.1. *Some fractal forms that make use of the principle of allometry*

Based on the works of B. Mandelbrot, the awareness and interpretation of the power laws is initially applied to geometric objects, forms and morphology (see Figure 1.1).

Of course, it is possible to find precursors to these studies. For example, the works of biologists on allometry have also applied to the differential evolution of some and all parts of the human body. The negative allometry between the head and body of a human being is well known. The head does not grow as quickly as the body; compared to an adult, the head of a baby is always bigger. Similarly, Kléber's law pronounces a dependency between base metabolism – the energy required for life at rest – and mass for various species. As with a fractal, allometry is modeled by a power law. These objects are rarely auto-similar; the head, for example, does not have the same form as the rest of the body.

Allometry laws have also inspired a number of urban studies [LEE 89], and in his work *Théorie Générale des Systèmes* (General Systems Theory), L. Von Bertalanffy [BER 93] indicated that social differentiation and the division of labor in primitive societies, as with the growth of towns compared with rural populations, follow an allometric formula.

More recently J. Zhang and T. Yu [ZHA 10] have established allometric relationships for a large number of socioeconomic indicators on a global scale, with the basic unit being the state. They systematically tested 104 variables recorded in 273 countries. All of these variables were linked by a power law in terms of area, then population and finally gross national product. Adjusting between different variables and surface areas of states practically always provides a coefficient of less than 1, with the exception of arable lands, and thus a reducing allometry. This rule signifies that the growth of political units in terms of size is not necessarily an indicator of economic efficiency. Size is not synonymous with growth or economic development.

Other increasing allometries link representative variables from economic innovation and population, with an allometry parameter greater than 1. Conversely, this parameter is less than 1 for energy consumption and infrastructure and close to 1 for residential supply and demand indicators. Sometimes the interpretation of these increasing or decreasing allometries remains incomplete. The authors of this study [ZHA 10] have, however, proposed an original idea that involved widening the allometry parameter calculated between two variables to be a ratio of two fractal dimensions.

Similarly, in 1940 Korcak [KOR 40] established a power relationship between the number of islands and their size.

R.H. MacArthur and E.O. Wilson [MAC 67] constructed the theory of the ecological wealth of islands by linking the number of species and the size of the island using an allometric equation.

In anatomy, as in hydrology, the power laws were very quickly applied to data representing the ramifications of a network.

Fractal geometry will generalize these approaches for most physical, living or social forms.

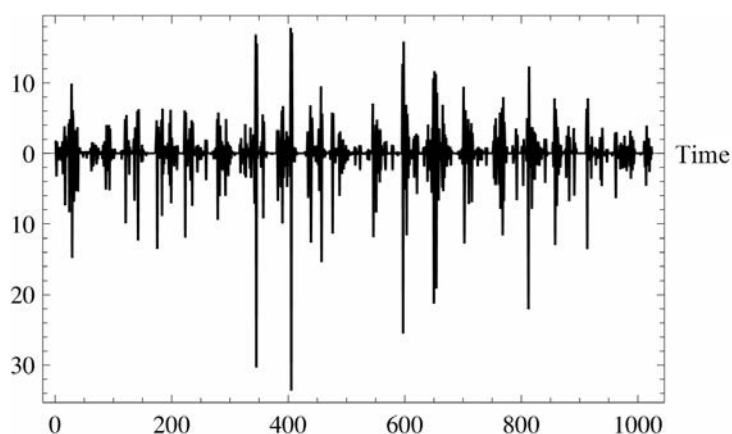
### ***1.2.2. Time series and processes are also fractal***

Virtually at the same time as the discovery of fractal forms was gaining momentum, fractal formalism was being applied to processes. These works have opened the way to a new interpretation of geophysical, particularly climatic phenomena, but also economic and financial mechanisms [LEV 02, MAN 97]. In these studies, fractal dimensions are directly calculated on chronological sequences.

At the beginning of these studies, L. Bachelier [BAC 00] emphasized the random nature of stock exchange series. In 1900, he advanced a Gaussian model for formalizing these evolutions. A number of works confirmed this model before B. Mandelbrot emphasized its limitations. L. Bachelier's model discarded extreme fluctuations, which were considered to be exogenous accidents, but B. Mandelbrot wanted to retain them and suggested that the Gaussian law should be replaced by a Pareto distribution. The abnormality of stock exchange series is now accepted, and the self-affine nature, illustrated in Figure 1.2 and inferred by the choice of a power law, is established.

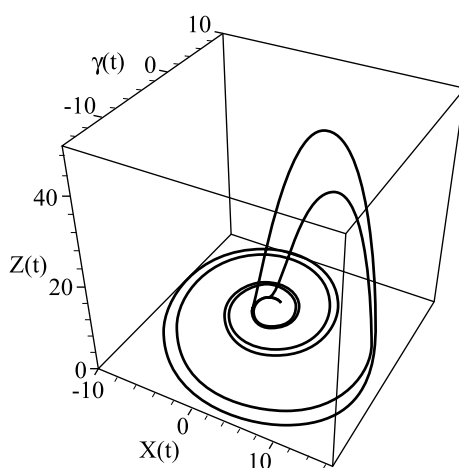
Explanations of stock market crashes have completely changed since an invariant behavior signifies that major crashes have the same root causes as minor fluctuations. Some new explanations have therefore been required. Crashes are endogenous and not provoked by external incidents. This principle has been generally applied to most catastrophes (see Chapter 9).

Additionally, studies into the dynamics of a phenomenon are often conducted from a phase space. Without going into detail, which would necessitate some long drawn-out dynamic systems theory (see Chapter 8), we recall that a phase space causes attractors that drive system behavior to appear.



**Figure 1.2.** *Self-affine time series*

Of these, strange attractors – which have been discovered recently – are indicators of deterministic chaotic behavior, even though they mimic random evolution. Yet, these strange attractors are fractals (see Figure 1.3). The fractal nature of an attractor is proof of a chaotic evolution. The Lorenz fractal attractor emphasizes the unpredictability of time beyond around 10 days.



**Figure 1.3.** *A fractal strange attractor*

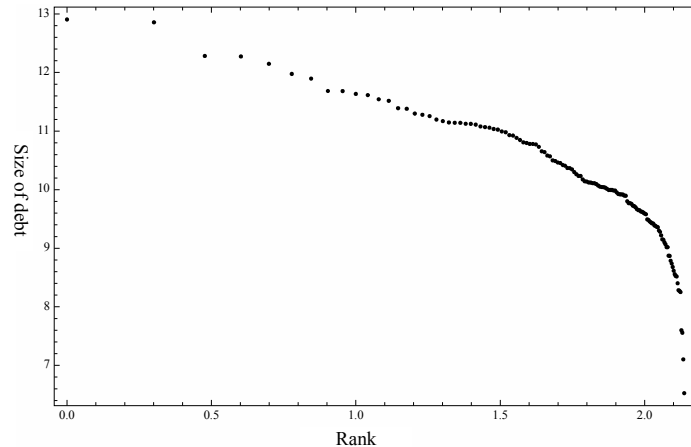
In addition to this, the Avignon geographers looked into transitions between fractal and non-fractal states, with their origins in the essential discontinuities that

shape our perception and representation of the world. Thus, fractal formalism has been extended beyond its initial scope of a simple morphological description of geographical objects. It is also applied to the dynamics of geographical phenomena.

### 1.2.3. Rank-size rules are generally fractal structures

Fractal reasoning has been applied to frequency series that are representative of functional structures. As a result, a number of scientists soon abandoned Gauss' normal distribution as a way of representing certain asymmetrical distributions. Some retained power or hyperbolic laws in order to adjust their experimental data. These often bear their names, such as the Richter-Gutenberg or Auerbach-Zipf laws.

Thus, in adjusting these data using a power law, geographers may look for signs of fractality, whether these are images, maps, time series or simple frequencies. Often it is sufficient to classify the intensity of observed phenomena based on their rank. However, as we shall see in Chapters 4 and 5, a number of precautions need to be taken. A visual examination of the information is definitely required, but this is not enough in itself. Figure 1.4 represents a series of the public debts of 273 states, which can be adjusted either by a power law or by a log-normal distribution.



**Figure 1.4.** *A functional fractal: the public debt of States*

### 1.3. First reflections on the link between power laws and fractals

In order to attribute the term fractal to a phenomenon, we must carry out a power law adjustment. It is, furthermore, necessary to determine that this geographical

phenomenon, which is represented by a form, time series or a simple frequency histogram, is multiscale, then that it is either auto-similar or self-affine. All fractals are expressed in terms of a power law, but the inverse is not true. This rule merits some attention.

### 1.3.1. *Brief introduction into power laws*

Like all statistical rules, a power law is represented either by a probability distribution (relative frequencies) or a distribution function (cumulative relative frequencies). For a probability density function (PDF), this is expressed as:

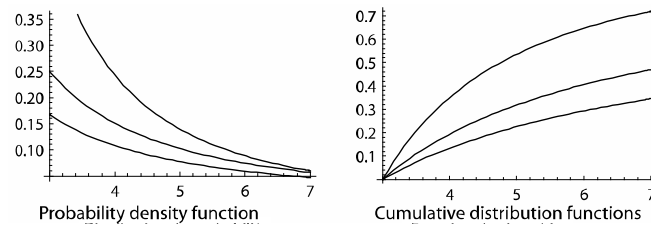
$$P(X = x) = ax^{-\alpha} \quad [1.1]$$

and for a cumulative distribution function (CDF):

$$P(X > x) = bx^{-(\alpha+1)} \quad [1.2]$$

A power law is characterized by its scaling exponent, or parameter,  $\alpha$ . The symbol  $\alpha$  was chosen with reference to the works by V. Pareto [PAR 97]. This scaling exponent should not be confused with a fractal dimension. The fractal dimension is effectively deduced from this exponent, but based on a formula that varies based on the fractal type being analyzed and the algorithm used for its calculation. This is the focus of Chapters 4 and 5. Many researchers have been confused by this and have occasionally advanced the notion of a fractal dimension that is less than the topological dimension.

The attentive reader will have noticed that the two exponents from the PDF and CDF are unequal, but directly linked. The value 1 must be added to the slope when the adjustment is effected based on the CDF in order to obtain the scaling parameter. In the same way that the fractal dimension is calculated from this exponent, which is also known as the power law scaling parameter, it is advisable to take care over the formalism used. Figure 1.5 depicts two frequential versions of power laws where exponent  $\alpha$  is below zero. When this exponent is positive, the PDF radically changes shape. In his original formulation, the Pareto distribution was represented by a CDF, i.e. cumulative frequencies, as Pareto wanted to study the behavior of revenues above a fixed threshold.



**Figure 1.5.** *The two faces of power laws*

However, power laws are rarely plotted on a graph where the coordinates are arithmetical. Whether we are referring to a PDF or a CDF, power laws are on the straight-line sections of a log-log graph. This property forms the basis of a visual test. Data positioned on the straight-line section of a log-log graph, if it is multiscale, intimates the existence of a power law, and therefore a fractal phenomenon. This test is not without its faults, however, as other laws, particularly the log-normal distribution, display very similar behaviors (see Chapter 5); and yet we should differentiate between them.

Occasionally frequencies are replaced by rank-size rules. Data classified according to their rank are equivalent to a distribution function, with one difference. Rank-size rules enhance big events, such as capital cities or earthquakes of exceptional magnitude. In representations of a distribution function, this is contrary to small events that are better displayed.

There is a new mathematical relationship between the three exponents of the Zipf, Pareto and power laws. If  $\beta$  is the Zipf law exponent, then the Pareto distribution exponent, which is equivalent to the PDF of the power law, is equal to  $1/\beta$ , and the corresponding CDF has an exponent equal to  $1+1/\beta$ . When the data follow a strict Zipf law, i.e. when  $\beta=1$ , the power law or scaling parameter exponent is equal to 2. These three exponents should not be confused, as the fractal dimension is deduced from the power law scaling parameter. The exercise is all the more difficult in that numerous generalizations of these elementary power laws are possible. Some of these are explained in later chapters.

A power law presents three complementary characteristics:

- First, it is the only statistical distribution not to have introduced a characteristic scale, temporal or spatial. It is therefore the only scale-invariant statistical law and is therefore representative of fractality.
- Second, when the value  $\alpha$  of the PDF of a power law is between 0 and 2, the variance is infinite. The same applies to the mean if  $\alpha$  is less than 1. Such values

indicate very high asymmetry, and particularly a higher frequency of extreme values relative to that observed under the normal distribution.

– Third, a power law is said to be heavy-tailed, which signifies that the correlation disappears slowly in time or space. Contrary to the exponential distributions (Poisson, normal or log-normal), which are without memory in time, a power law presents a strong auto-correlation. The persistence of a significant correlation is therefore an indication of fractality. However, we are referring to an indication and not proof. All power laws are heavy-tailed, but the opposite is not true. A wide-tailed distribution is not necessarily an indication of a power law. Nevertheless, in all of these distributions significant events have an essential weighting, which influences the values of mean and variance.

### ***1.3.2. Some power laws recognized before the fractal era***

Knowledge of power laws precedes that of fractals. Galileo was already making use of a power law in order to explain why trees did not reach up to the sky. The Gutenberg-Richter law in seismology, the Pareto distribution in economics and the Zipf law in linguistics also anticipated the fractal revolution. However, remember that not all power laws are, by default, fractals. A fractal is observed over a wide range of levels; it is a multiscale object. The power law is a necessary condition of, but insufficient to define, a fractal. Most homogeneous power laws are not multiscale and so are not therefore fractals.

Similarly, power laws that describe the surface of a square or a cube, with a whole-number exponent equal to 2 or 3, are not fractals, even though the exponent is an indicator of the form of these Euclidean objects. However, this Euclidean dimension gives no information about the irregularity of this form. A line, which may be straight, curved or very broken, always has a Euclidean dimension of 1. Conversely, the fractal dimension of a straight broken line, which is made up of many segments, is greater than the fractal dimension of a virtually rectilinear line. This fractal dimension is all the more considerable when the line being considered is irregular. Other power laws, which are more commonplace in economics, have been interpreted in terms of elasticity. Insofar as elasticity is defined as a relative variation ratio between two identifiers, the slope of the log-log graph is comparable to a measure of elasticity. Take the example of a Cobb-Douglas function written in the form:

$$P = aC^{0.65}T^{0.25} \quad [1.3]$$

where  $P$  is production,  $C$  is capital and  $T$  is work.

This law simply signifies that a 10% increase in capital leads to a 6.5% growth in production, and if the work factor increases by 10%, then production rises by 2.5%. This is certainly a power law, but is not a fractal, since it lacks scale invariance.

Above all, one of the points of using these power laws was to compare the asymmetry of a statistical distribution relative to Gauss' normal and symmetrical distributions. The Pareto distribution was moreover described in terms of the 80–20 rule, as throughout, household wealth was and remains the focus within a number of families. The focus was not on auto-similarity or scale invariance but on the weighting of significant events and on extreme values. In climatology and hydrology, power law adjustment was a classical routine, as was proved by the school works of C.-P. Péguy and our own thesis, but without leading to a genuine reflection of scale invariance.

An identical report was applied to studies of urban hierarchies conducted during the 1960s based on the Auerbach-Zipf model. These were not the subject of fractal interpretations. To confirm this, we only have to read the works of B.J.L. Berry [BER 64] or those of specialists in French urban geography [BEA 97, CLA 81]. For an urban hierarchy, these authors attached great importance to cathedral cities, such as London, Paris and Beirut, and therefore once again to extreme values.

Specialists in urban geography were not focusing their attentions on the scale invariance of phenomena. They continued to apply classical statistical reasoning by insisting on the asymmetry of a distribution relative to the normal distribution. In this light, B. Berry [BER 64] gave a ranking to commercial centers using this reasoning. Then in France during 1957, in his statistics course, C.-P. Péguy [PEG 57] illustrated the urban hierarchy of Brazil using a Pareto distribution to emphasize the weighting of large agglomerations in this new country. The scale invariance of urban hierarchies is obviously a focus of research and has been dealt with in many articles over the past 10 years [DIM 07, SCH 09], but is still ignored by manuals on urban geography.

In the recent volumes on French towns, written in preparation for aggregation examinations, the focus was still on the primacy factor, i.e. the weighting accorded to Paris. All of the authors of these volumes devoted a chapter to this issue, but no practical mention was ever made of the scale invariance of the French urban network and its fractality was therefore ignored.

In reality, whatever the theme tackled, it was a question, above all else, of better appreciating or predicting significant events. Indeed, the power law attached great significance to great wealth, and therefore to extreme events that had a strong influence on the average. All statisticians know that there is virtually no chance of finding a soldier who is twice as tall as one of average height; he would have to be

over 3.4 m tall. Soldiers' heights follow a normal distribution. Conversely, the same reasoning applied to taxation which indicated that around 7% of people had paid more than double the average in taxes. This approach was therefore generalized in the study of risks, both financial and natural. We will develop this theme in Chapter 9.

#### **1.4. Conclusion**

In conclusion, since the pioneering works of A. Dauphiné and P. Frankhauser on French geography and of P.A. Burrough or S. Lam and L. De Cola on US geography, fractal analysis has been strongly enriched and diversified. Aside from subsequent research from geographers abroad, studies making use of fractal and multifractal formalism are becoming increasingly commonplace, since most geographical phenomena are coarse, irregular and multiscale.

Furthermore, the fractal approach is no longer summarized by a simple calculation of the fractal dimension of a surface object, irrespective of whether it is a mountainside, vegetation cover or an urban area. The fractal approach can also be applied to processes and mechanisms that guide behavior and the evolution of territories probed by geographers.

Finally, although this presentation on power laws has been very brief, it should be clear to the attentive reader that the calculation of a fractal dimension, which depends on the scaling parameter, can be performed using different techniques. We will return to these regularly over the next five chapters.



## Chapter 2

# Auto-similar and Self-affine Fractals

“Fractal theory is a theory of the broken, fractured, scattered or even the granulated, porous, tangled, etc.”

Alain Boutot, 1993

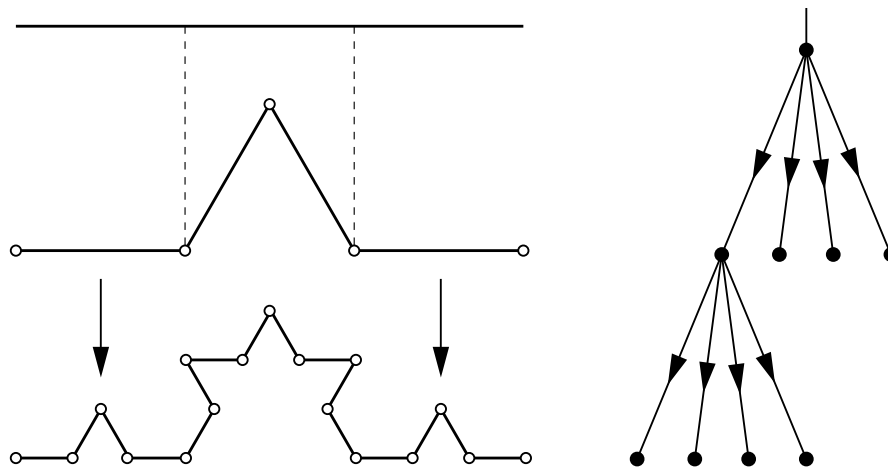
From his first works, B. Mandelbrot insisted on the ubiquity of fractal forms by exposing numerous examples. This diversity has grown with the fractal treatment of chronological series, which are representative of processes; and by numerous methodological and technical extensions. Faced with a profusion of fractal forms and processes, it is natural to imagine a typology and to distinguish multiple categories that are observed to a greater or lesser extent. This classification is all the more essential when the calculation of the fractal dimension, which is developed in later chapters, depends on it.

This chapter therefore classifies geographical phenomena according to their pertinence to a type of fractal. Here we will distinguish auto-similar from self-affine fractals. Then, in Chapter 3, we will refer to the multifractal approach, which extends fractal formalism to sets described according to a measurement, for example a digital terrain model, a town shown in grayscale as with satellite images, a chart showing rainfall levels or population densities, etc.

### 2.1. The rarity of auto-similar terrestrial forms

The first fractal forms – which were seen in the 19<sup>th</sup> Century and described as “mathematical monsters” by their discoverers but called fractals by B. Mandelbrot – were geometrical concepts. They were applied to strictly auto-similar material objects. These first fractals studied by mathematicians, such as the Cantor set, the Peano curve or the Sierpinski triangle, are of historical interest since fractal geometry is constructed by analyzing their properties. They have helped in the study of not just the geometry of spatial forms and their irregularity on multiple scales, but have also enabled a better understanding of the concept of auto-similarity.

These auto-similar fractals, which perfectly illustrate the various volumes dedicated to fractals, remain good educational examples. They show how a simple rule can create a complex and irregular fractal form (see Figure 2.1). In order to draw a fractal, we need an initiator, such as a line, and then a generator, such as a triangular-shaped fragment, and finally an iteration. An auto-similar fractal is obtained through several iterations of a generator onto an initial “figure”. In the example shown in Figure 2.1, each segment is decomposed into four more segments at each iteration. Successions of branches in a river network, a tree or in a military organization result in the emergence of fractal forms.

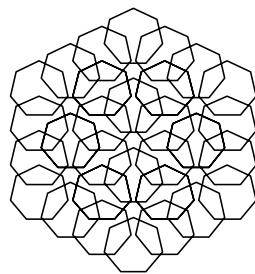


**Figure 2.1.** *Creation of an auto-similar fractal by iteration*

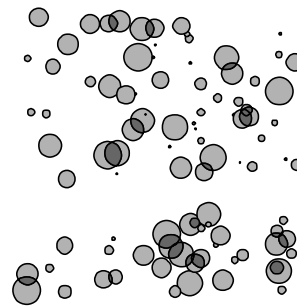
By varying the first two elements, and by modifying the number of iterations, it is possible to design an infinite number of very complex forms. For example, the generator is often deterministic, but it may also include an element of luck. With

luck, the figures obtained from multiple iterations depict stratified random fractals, whose forms are closer to natural forms than deterministic fractals for which irregularity, which is too structured, is also too predictable. Figure 2.2 illustrates two fractals drawn from initiators and generators that are more original than just simple lines. They are reminiscent of territorial organizations that are more realistic than geographers may encounter.

Finally, complexity is derived from simplicity. Behind all of the irregularity and coarseness some simple mechanisms are hidden. Fractals do not glorify the complex, but do help to unravel this complexity and to interpret it using elementary and repetitive mechanisms on different levels.



Fractal polygons



Stratified fractal

**Figure 2.2.** *Realistic auto-similar fractals (source: Mathematica Demonstrations Projects, Sandor Kabal, Stephen Wolfram)*

Despite their diversity, for most disciplines, including geography, auto-similar fractals present three handicaps. First, auto-similar forms are rare. On land, as with all simple Euclidean forms such as the circle or square, auto-similar geometric forms are the exception. Most of the time, auto-similar forms are deformed in one or more directions in space. These are therefore more complex self-affine forms rather than auto-similar objects. In the “real” world, more often than not auto-similar objects are very small in size, such as a snowflake, which is subjected to few physical constraints. These are not really of interest to geographers.

Second, whereas the fractality of mathematical objects has no boundaries, material fractals have upper and lower limits. A cauliflower, for instance, does not have an infinite size and its fractality does not drop below a certain threshold. This rule applies to all geographical phenomena. B. Mandelbrot even proposed that infinitely auto-similar mathematical fractals should be distinguishable from the prefactals universally encountered, but this term has not been adopted by the

scientific community. These two proper limits should not be confused with any non-mathematical fractal, with the resolution and range that is therefore dependent on single observation of data handled. These two types of limitations rarely coincide.

Third, certain fractal dimension calculation techniques are only valid for this one category of auto-similar forms. This is the case for the majority of dimensions calculated using a technique known as box counting, or paving stone, which are explained in detail in Chapter 4, and yet many geographers still favor these techniques for analyzing forms that are self-affine, rather than auto-similar. This is often the case in urban geography studies, although J. Russ [RUS 94] had highlighted this mistake in a well-documented publication. This error was well explained and corrected in some highly technical articles and L. Seuront's recent publication [SEU 10].

## 2.2. Yet more classes of self-affine fractal forms and processes

Most "natural" or social terrestrial forms are in fact self-affine or multifractal forms. Before tackling the latter in Chapter 3, we should focus our attention on self-affine forms that frequently occur in geography. As we emphasized in a previous publication [DAU 95], scale invariance, which is rarely isotropic, is often deformed in a single direction. In this instance, a part of the object is no longer representative of the whole, even though it possesses the same statistical properties on all levels.

For example, gravity provides an original weighting to the vertical axis of a relief, relative to the two horizontal axes. These three axes are not the same in nature. Longitude and latitude are only useful reference points for localizing altitude. This gravitational effect is significant in all natural phenomena, in meteorology as much as in ecology. The term auto-similar is therefore unsuitable and the object being studied is a self-affine form. In another vein, a valley such as the Seine directs the urban spread of a great metropolis, such as Paris, and the development of coastal towns is limited by the shape of the coastline.

Recently, Y. Chen and J. Li [CHE 09] demonstrated that every town is a self-affine fractal. They based this demonstration on the allometric distributions between the spread of built-up areas and population of each town, and on the Cobb-Douglas power function. This demonstration, the explanation for which is outside the scope of this book, confirmed the empirical observations of the classical geographers, who insisted on extending towns in different directions, and contradicted the hypothesis of an auto-similar town.

These self-affine forms can be observed everywhere: clouds blown by the wind, reliefs structured along major axes and hydrographical networks are all self-affine

fractals. However, this type of fractal is also a characteristic of most man-made forms, such as road networks, the internet, and town morphology, such as that of urban networks. Practically all landscapes are anisotropic. They are subject to the effect of various gradients: pressure or gravity for “natural” spaces, distance gradients, cost and segregation for humanized spaces. This rule has been accepted for as long as geography has existed, even if, for convenience, theoretical analyses have departed from homogeneous and isotropic spaces.

Additionally, all chronological series are self-affine fractals, whether they are precipitations, graphs showing frequencies of theater visits or stock market trends. Indeed, as is the case with reliefs, the x axis representing time and the y axis representing the variable studied are not of the same nature. It is easy to lessen or enhance the impact of evolution by varying the position on the y axis. Politicians use this in order to make a crisis appear or disappear, or to increase or decrease the visual impact of unemployment graphs. In reality, all processes that are illustrated through chronicle series are self-affine.

When conducting an observation, such as a pluviometric series or satellite image observation, the first step is therefore to identify which type of fractal is relevant; is the fractal auto-similar or self-affine? Technically, this is determined using a visual test, a Fourier or a wavelet decomposition or, even more unusually, from an analysis of trend fluctuations. Some of these techniques are presented in detail in Chapter 4.

This first stage is imperative, since the choice of fractal dimension calculation algorithm depends on it. Some techniques only make sense for auto-similar fractals, while others are only valid for self-affine fractals. Furthermore, there are several main categories of self-affine forms, and yet each subclass of self-affine fractals requires appropriate fractal formalism. It is therefore advisable to spend some time on these distinctions within the category of self-affine fractals.

### **2.2.1. *Brownian, fractional Brownian and multi-fractional Brownian motion***

In order to understand the diversity of self-affine fractals, it is useful to depart from formalized diffusion using the Brownian motion. As with auto-similar fractals, all of the motions described in this chapter are equivalent to iterations applied to these generators, rather than elementary displacements. The consequences of these iterative displacements in time and space create fractals.

### 2.2.1.1. *The two fractal facets of Brownian motion: auto-similar in space and self-affine in time*

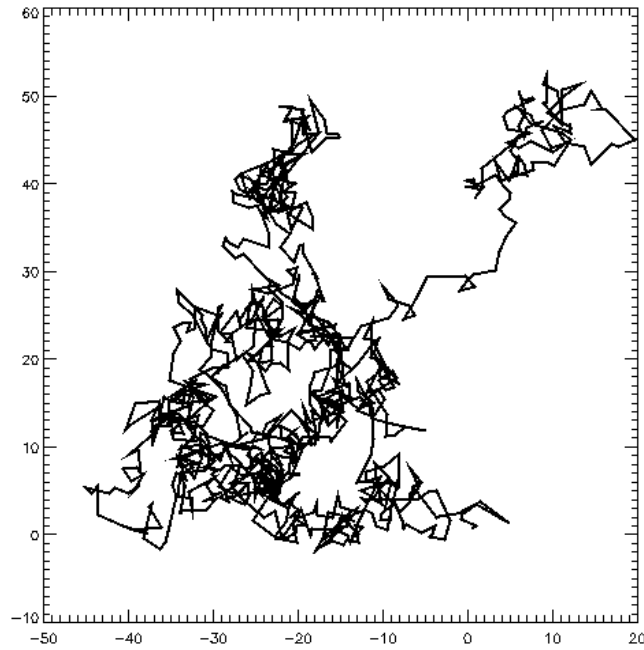
For physicists, diffusion comprises microscopic motion provoked by thermal agitation. By observing the diffusion of pollen grains in a liquid, the biologist Brown showed that their diffusion could not be attributed to life. Einstein gave the theoretical explanation of this motion, which was the direct result of thermal agitation – a theory that was validated by Perrin's own findings. Nowadays, by analogy, diffusion describes all motion arising from very short, successive displacements. For example, pollutants are diffused in a lake, and a rumor gradually spreads inside a concert hall or around a town.

Yet, diffusion was presented by numerous models, both macro- and micromodels. It was first mathematized by Fick's deterministic macromodel, which was put forward in 1855. This model generalized the heat diffusion equation established by Fourier.

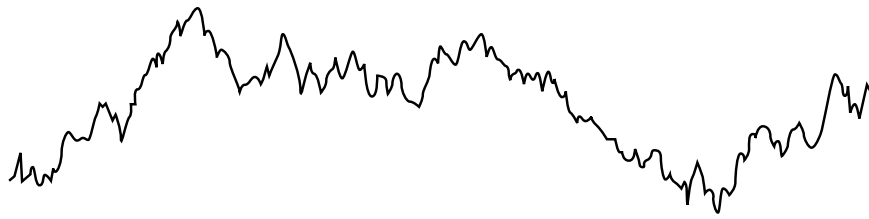
Later, diffusion was modeled by random particle movement. Normal diffusion is thus simulated using a displacement, which is random in two respects – direction and intensity – but with a finite mean. The intensity of successive increments,  $X_{(t+h)} - X_{(t)}$ , obeys a Gaussian null mean distribution with a variance equal to  $h^{2H}$ , where  $H$  is equal to 0.5. Described as white noise, Brownian motion is an illustration of a mild collision as a result of a normal distribution based on B. Mandelbrot's imaged expression. In Brownian motion, very large displacements are extremely rare. In a Gaussian distribution, the mean rightly corresponds to the most frequent events. Brownian motion materializes in time and space as a result of Gaussian distribution [HER 10].

A significant distinction should be established between Brownian particle motion in space, or trajectory, and its motion in time, which is described as Brownian trace or function. In space, the trajectory of Brownian motion links the extremities of successive displacements. This trajectory or streak is an auto-similar fractal whose fractal dimension is equal to 2, since for an infinite number of points the lines that define this trajectory cover the whole space. Figure 2.3 illustrates this motion well before infinity is attained. By zooming in on this figure, as with all auto-similar fractals, we would obtain the same figure.

In time, motion is illustrated by mean quadratic displacement, not by null mean displacement (see Figure 2.4). A mathematical trick to help overcome the problem of null mean displacement is to think of mean quadratic displacement as being equal to the square root of the square of the mean displacement. This motion in time is described as a trail or graph. It varies according to a power law for which the fractal dimension is no longer equal to 2, but to 1.5.



**Figure 2.3.** *Brownian motion in space*



**Figure 2.4.** *Brownian motion in time*

This value is directly linked to the Hurst coefficient,  $H$ , which was elaborated by a hydrologist who studied Nile flood waters. This scholar's objective was to choose the size of dam that would ensure that this river's flood waters were regulated, so as to reduce the risk of famine in Egypt. He proposed a calculation technique called rescaled range analysis in order to determine the value of parameter  $H$ . This approach was not built around the intensity of variation, but on its temporal organization. For simple Brownian motion, which is synonymous with normal diffusion, the Hurst coefficient,  $H$ , is simply equal to 0.5.

Such a value indicates the absence of auto-correlation in the time between successive increments  $X_{(t+h)} - X_{(t)}$ , and is testimony to the Markovian characteristic of Brownian motion. This mathematical expression indicates that the size of an increment is independent of the size of previous increments. However, note that, conversely, successive values of  $X_{(t)}$  are strongly auto-correlated.

This temporal distribution is self-affine, rather than auto-similar. Its fractal dimension is 1.5, i.e. equal to  $2 - H$ . In this case, 2 corresponds to the topological dimension of a line that represents the chronicle series, to which a value of 1 can be added. The curve in Figure 2.4 displays a “mean” irregularity that is repeated for all temporal levels. It is therefore self-affine.

Brownian motion has therefore spawned two fractals: a form that is fractal in space and a temporal fractal whose dimensions are different. It is important to avoid any confusion between these two fractals, and this duality, established for Brownian motion, remains valid for all of the motion explained below.

Brownian motion was used by L. Bachelier [BAC 00] in order to represent the evolution of French treasury bonds, although they were not called this at the time. He chose this law because he estimated that this evolution was the result of a multitude of small reasons, which is the main feature of a normal distribution. Gaussian distribution will need to be revisited in the vast majority of studies related to finance before B. Mandelbrot’s critical analysis can be accepted by specialists from the financial community [MAN 97]. It may take crises on the scale of those following the “Thirty Glorious Years” before the Gaussian model can be called into question again.

Brownian motion, which is the equivalent of a normal diffusion, is rarely encountered in nature and even less so in humanized territories. Diffusion is often agitated or accelerated by innumerable mechanisms. For example, pollutants are diffused more rapidly in an animated expanse of ocean current or in air that has been agitated by wind, such as the example of the radioactive cloud from Chernobyl. Blown by exceptional easterly winds, the radiation from Chernobyl threatened Northern and Western Europe. Conversely, in a metropolis the movement of people is impeded and slowed down as a result of a number of obstacles: buildings, railway lines or rivers in estuary towns such as Bordeaux or Nantes. Similar impediments occur in natural phenomena.

Long displacements may also occur and complicate the appearance of normal diffusive motion. Thus a businessman may cause multiple diffusive displacements within a metropolis, but these are interrupted by long-distance journeys to other cities around the world. From this fact, in the real world, innumerable factors may disrupt normal diffusion. Several models have been constructed in order to represent

these abnormal diffusions. Geographers, who compared diffusion by contiguity with hierarchical diffusion while jumping from city to city, participated in this work [SAI 85].

Indeed, in nature as in society, the two fundamental hypotheses of Gauss' law – the size independence of increments of change and the absence of large gaps – are rarely verified. Whether we think of movement in share prices or changes in temperature or rainfall, we observe correlations over time, not only between these variables but also between their increments. This is even truer with social phenomena, where the effects of memory and learning are always present. This invalidates Gauss' law and results in forms of abnormal diffusive motion. This type of motion is more common.

#### 2.2.1.2. *Fractional Brownian motion and fractional Gaussian noise*

Unlike geographers, physicists distinguish not just two, but an infinity of diffusive motions. They offer a truly vast range of models that yield yet more fractal forms. The first, a classic example, creates self-affine forms generated by a fractional Brownian motion, or fBm. This initial generalization of Brownian motion was proposed by B. Mandelbrot and J.W. Van Ness [MAN 68].

In a fractional Brownian motion, the Hurst coefficient remains constant, while not being equal to 0.5. Certainly it keeps a unique value, but this is between 0 and 1. While moving away from the value 0.5, the rise in the displacements is no longer independent, as with normal Brownian motion, but is correlated with them. When the Hurst coefficient is above 0.5, it translates into a persistence or sub-diffusion effect between successive displacements, while an anti-persistence or super-diffusion effect is obtained with a Hurst coefficient below 0.5. In this type of motion, the trail tends to return to the point where it came from. In space, therefore, diffusion is slackened.

In time, the persistence effect indicates that low values are followed by low values and high values by high ones. Persistence series have a long-term memory. At time  $t$ , the state of the system is dependent on the whole life of the system, and therefore on its earlier states. Conversely, anti-persistence is synonymous with great irregularity, since low values follow high values.

For example, the annual flooding of the Nile, before the construction of the Aswan dam, had a Hurst coefficient of 0.74. Therefore, there tended to be successive years of either drought or good rainfall, which then caused years of famine followed by years of abundance. This persistence was described as the Joseph effect by B. Mandelbrot, in reference to the biblical story. Various authors have commented that “natural” phenomena have a value of  $H$  approaching 0.8, which is testimony to

a diffusion effect that is accelerated as much in time as in space. However, Hurst coefficient values above 0.7 are also common in studies of car traffic flow.

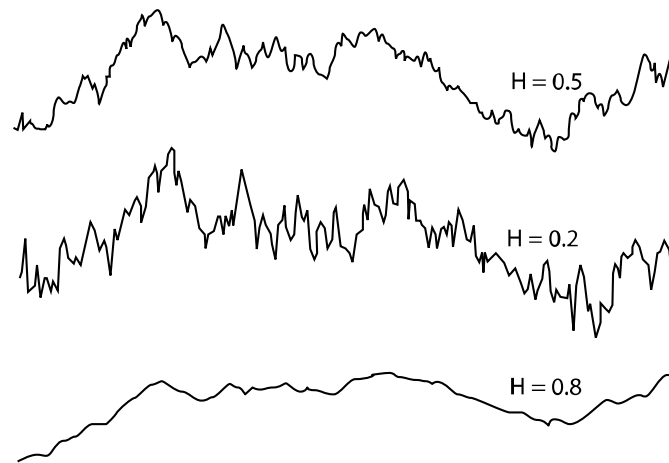
Following on from the work of B. Mandelbrot, fractional Brownian motion was the subject of many works in financial economics. Some verifications have relied on the changes in various actions and also on the rate of change – the value for  $H$  was almost always different to 0.5. The Bachelier model and the more complex models that followed this, while still retaining a Gaussian schema, continued to be taught in the more prestigious colleges before being discounted following the crises of the past 20 years. For example, according to a Gaussian evolution model, a 6.8% fall in the Dow Jones, as happened on August 31, 1998, would have been unlikely to have occurred once in every 100,000 years [HER 10].

This type of reasoning has also been applied to climate change by A. Hussein and R. Ansari [HUS 06]. Hurst coefficients equal to 0.40 and 0.31, which were calculated based on thermal series of 1860–1914 and 1915–1954, respectively, remained below 0.50 and indicated the absence of a persistence effect and therefore a tendency towards cooling or warming. Climate was not subject to any trend changes during these two periods. Conversely, during the 1955 to 1999 series, the Hurst coefficient was equal to 0.54. High temperatures were thus followed by more high temperatures, which translated into a confirmed warming of the overall climate. This persistence was relatively weak, however, since the value of the Hurst coefficient was very close to 0.5. Furthermore, in this case it would be advisable to consider other temporal ranges.

Figure 2.5 illustrates some one-dimensional fractional Brownian functions. It is easy to see that a high value of  $H$  (0.8) is synonymous with low irregularity, relative to a curve constructed with a low Hurst coefficient value (0.2).

In two dimensions, numerous relief simulations have also been produced with a fractional Brownian motion. By modifying the Hurst coefficient value it is easy to accentuate topographical irregularities. In the virtual world of cinema, for example, some striking visual effects have been executed by instantaneously modifying the value of  $H$ .

Fractional Brownian motion has another essential characteristic. It is non-stationary, which means that the mean and/or the variance increases according to the length of a series or the size of space considered. This increase is based either on the Joseph effect, and therefore a slow drift linked to persistence, or on the appearance of extreme values, described as the Noah effect as a tribute to the floods that covered the Earth.



**Figure 2.5.** *The Hurst parameter,  $H$ , which is indicative of Brownian motion irregularity*

Fractional Gaussian noise (fGn) is a second source model of self-affinity. Although derived from the preceding model, it possesses singular characteristics.

First, it is stationary, the consequences of which have a twofold importance. On one hand, extreme values, and therefore the risks associated with them, have a totally different behavior. They are much more common and much less predictable in fractional Brownian motion than in fractional Gaussian noise.

Second, even if the Hurst coefficients,  $H$ , are equal, the equations used to calculate the fractal dimension are dissimilar (see Chapter 4). It is therefore necessary to know the motion corresponding to the data studied in order to avoid the calculation errors and fanciful interpretations that are inherent in all disciplines.

Finally, fractional Gaussian noise is self-affine in time but, unlike fractional Brownian motion, it is not auto-similar in space.

#### 2.2.1.3. *Multifractional Brownian motion and Gaussian noise*

These two generalized motions were the subject of a new generalization, notably in financial economics and morphometry. We therefore refer to multifractional Brownian motion or multifractional Gaussian noise. Within these models, the Hurst coefficient,  $H$ , is no longer a constant with a value between 0 and 1, but becomes a variable,  $H(t)$ , which is a function of time for a series or space for a digital land model. In practical terms, this generalization enables different persistent or anti-

persistent behaviors to be attributed to parts of a chronological series or of an image of a relief, for example.

Let us again consider the example of a relief simulated using a fractional Brownian motion. By reducing the Hurst coefficient, the whole of a relief fashioned from a digital land model becomes more irregular. This increasing irregularity affects plains as well as mountains. It is global and yet, in reality, the irregularity is stronger in mountainous areas, where altitude increases the effects of erosion. In order to better represent this constraint, a multifractional Brownian model is therefore more effective than a fractional one. J. Lévy-Véhel [LEV 03] has suggested that this is enough to control the Hurst coefficient according to altitude. At higher altitudes, the value given to  $H$  will be below 0.5, but will be higher at low altitudes. The landscapes obtained in this way are very realistic, at least for small-scale reliefs. Indeed, this generalization is still insufficient to represent the reliefs of vast mountains, such as the Himalayas or those in Tibet, although the simulations of some parts of these vast mountains produced with a multifractional Brownian model do conform with reality.

This type of modelization was adopted by P. R. Bertrand *et al.* [BER 10] in order to successfully analyze the fractal nature of NASDAQ prices from 1971 to 2009. However, these authors chose a semi-affine function in order to control the value of parameter  $H$ . This function, formed from line sections rather than a constantly changing curve, is therefore simpler, which is why they described their multifractional Brownian motion model as being parsimonious. The calculations were performed using FracLab software, programmed by researchers at INRIA.

Multifractional motion belongs to the class of multifractals. They cannot be described or simulated using a single Hurst coefficient, but require several. Different fractal dimensions correspond to these successive values for  $H$ . They are therefore clearly multifractals, which are presented in this chapter in the interests of educational convenience.

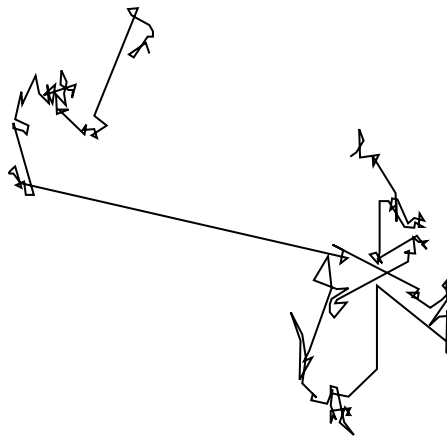
### 2.2.2. Lévy models

Lévy models make up another class, which is vast and quite different from that of Brownian motion. In a Lévy model, the direction of displacement is always random, as with a Brownian model, but the length of these displacements follows a power law rather than a Gaussian distribution. This is an illustration of B. Mandelbrot's beloved wild randomness. The rare events and the large displacements, which are the exceptions in a simple Brownian model, are more common and have a major influence on the mean displacement value in a Lévy

model. Based on the value of the exponent of the power law, we refer to a Lévy flight or walk [LAG 03]. These two subclasses are singular.

Lévy flights presume that each jump has the same duration, which is a little unrealistic if they do not have the same length; while Lévy walks introduce a variable time, which is based on the length of each jump. The longest jumps take the longest times. Lévy jumps are Markov processes, but this is not true of Lévy walks. The auto-correlation of a Lévy walk reduces more slowly in space or time than in a Gaussian Brownian motion.

In space, however, Lévy flights and walks combine in the form of short displacements, interrupted by long motion (see Figure 2.6). These models represent hierarchical diffusion, coupled with diffusion by contiguity, fairly well. This was used by geographers in order to explain dissemination from the mode or the projection of new films in the great metropolises and around these capital cities. They proposed a geographic model enabling the classification of motion that combines various forms of diffusion.



**Figure 2.6.** *Lévy motion in space*

In addition, a Lévy distribution conforms more to major crisis situations, with long periods of calm interspersed by sudden jumps, such as in a volcanic eruption or a stock market crash. These motions very faithfully simulate the intermittences that are the prerogatives of turbulent movements, which are very much a part of all natural (atmospheric currents, Gulf Stream, etc.) but also man-made phenomena.



**Figure 2.7.** *Lévy motion in time*

Several studies have confirmed that earthquake epicenters are determined according to Lévy flights, for example, around the Mediterranean. Recently, these Lévy models, which belong to the family of  $\alpha$ -stable laws, have tended to replace the Bachelier or ARCH (AutoRegressive Conditional Heteroskedasticity) Gaussian models in financial studies [LEV 02]. They have enabled good simulation of the behavior of financial markets, i.e. long periods of calm interrupted by brief periods of extreme volatility. Thus, for a number of observers, fluctuations in the CAC 40 of the Paris Stock Exchange are indications of a Lévy fractal.

In another domain, when food is less abundant, some ecologists have remarked that many predators, such as cormorants, fish according to a Lévy flight. They do not hesitate to fly long distances in order to discover new supplies of fish, which they may do many times. Some ecologists have even suggested that this fishing is optimal when the power law exponent is close to 2. However, these results remain a subject for debate [VIS 96].

More generally, in space Lévy motion produces aggregated clusters within aggregated clusters, as shown in Figure 2.6. They have therefore been used to simulate the creation of the universe and the emergence of galaxies. Various astrophysicists have shown that the paths of Oort cloud comets follow a Lévy walk.

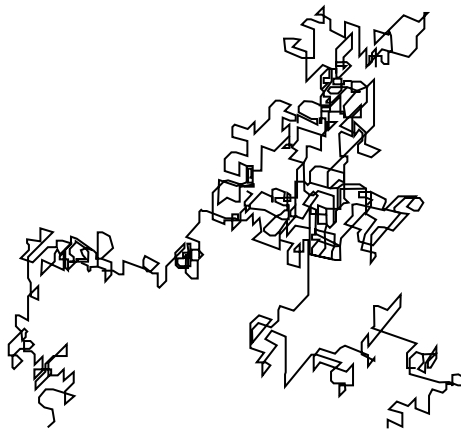
Finally, as with Brownian motion, we must distinguish forms in space, which B. Mandelbrot called Lévy dust, and graphs called Lévy or the devil's staircase. Lévy dust appears in *Les Objets Fractals* [MAN 75]. For Lévy staircases, the Hausdorff fractal dimension is equal to the topological dimension, i.e. 1, when referring to a chronicle series, which is proof that a fractal may, in exceptional circumstances, have a whole-number dimension value in spite of its irregularity. Additionally, probabilists are beginning to recognize multifractional Lévy motion although, to our knowledge, it is not yet the subject of any application in the social sciences.

Thus, without going into detail about fractal dimension calculations, which we cover in later chapters, some specific formulae have been considered for each model, which has reinforced the necessity of indexing the class to which the data being examined by geographers belong. Note also that most of these models have been adapted to sub-diffusive displacements, which are synonymous with long-term auto-correlation or space.

### ***2.2.3. Four examples of generalizations for simulating realistic forms***

These first simple Brownian motion generalizations remain generic models. However, in order to better comprehend concrete forms or processes, scientists have included a number of constraints in most of these generic diffusion models. It is not feasibly possible to set out all of these derived models, but four of them correspond to situations that are frequently encountered by geographers, and so merit a brief explanation.

The first example, self-avoiding motion, is used to model situations where no displacements return through the same point (see Figure 2.8). These processes are widely studied by polymer specialists. The fractal dimension of their trail is less than 2, which is the dimension of fractional Brownian motion, as they do not extend across the whole of the space plane. These models are also impacted by W. Tobler's first law of geography, that no two people or objects can occupy the same space at the same time.



**Figure 2.8.** *A self-avoiding motion model*

Thus, in a simulation model of urban growth, no house is built on an already developed plot of land, except in the case of previous destruction. In order to effect self-avoiding displacements, all previous motion needs to be remembered. This requires a complete memory, so as not to be confused with the following case, which is no less often encountered in geography. Some small examples of interest on self-avoiding motion can be found in the *Demonstration Projects* section of the *Mathematica*<sup>1</sup> website.

The second example, distinct from the one above, involves the displacement of many mammals, including man, through retained memory. They return to original sites, such as in the case of a pair of birds returning to a nesting place or humans returning to Venice or some place close to their hearts. New models, which include memory cards that direct these displacements, complicate Brownian or Lévy displacements. Some are made randomly, while others are set according to a saved objective.

Thus, repeated displacement of a university student towards his or her campus is the result of a memory effect, but occasionally they may stop to buy some cigarettes or gasoline on the way. Such stops are random sources of deviation. On this basis, recent studies, which have again formed the subject of debate, have confirmed that commonplace human displacement, which can be analyzed based on the locations of cell phones, follow a truncated Lévy flight.

Formalization of these models is currently being accomplished with the help of multisystem agents that provide great flexibility in the integration of these memory effects. A number of ecologists are working on models of regularly practised displacement. Software has even been written in order to simulate this motion [GAU 10]. However, abnormal diffusion, as represented by these displacements, often dictates a multifractal approach.

A third generalization is aimed at motion over a fractal space. In environmental geography, river pollution is directed by the fractality of turbulence, which drives water currents, and the river network. In large metropolises, displacements are also dependent on the fractal distribution of roads and train stations. D. Ben-Avraham's book [BEN 00] explores a number of examples of Brownian displacements over a fractal spatial structure. His work is very enlightening, as geographers are often confronted with mobility over fractal territories, not just in towns.

Finally, so we can better consider actual constraints, some rules relating to Brownian or Lévy model are occasionally modified. In order to illustrate the diffusion of a pollutant in a river, a drift effect produced by the current flowing

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<sup>1</sup> <http://demonstrations.wolfram.com>.

down the river should be added to the Brownian diffusion. These models are said to have directionally biased Brownian motion. It should also be highlighted here that multiple motion types may follow on from each other in time and space. For example, sand particles deposited on the surface of a river bed are subject to different types of transportation, either sub-diffusive or super-diffusive, depending on the stretch of water.

In another area of knowledge, financial economists have introduced quite irregular jumps within Brownian and Lévy modeling. These abnormal diffusion models often bear the name of their creator, for example the Morton and Kou models, which are described in the *Demonstration Projects* section on the *Mathematica* website. On the same subject, during diffusion certain particles are fixed onto a support for some length of time. These practices, which usually apply to humans, necessitate new diffusion models simulating these permanent or semi-permanent pauses.

In conclusion, the list of self-affine fractals is not a closed list. More are added to it every year. The majority of these models simulate super-diffusive, and therefore accelerated, behaviors. In order to characterize a phenomenon for which dependence is no longer long but short term, and therefore sub-diffusive, geographers make use of the so-called Ornstein-Uhlenbeck model [LI 10].

New techniques have constantly been developed in order to calculate the fractal dimension of this motion, since there is no unique formula for determining the fractal dimension of self-affine objects or series, which are the most frequently encountered in social sciences; and again, when calculating the fractal dimension of these processes, certain techniques have their drawbacks, particularly the paving stone technique, which is so widespread in geography.

### 2.3. Conclusion

Remarkably, geographical spatiotemporal phenomena are auto-similar fractals. Most auto-similar fractals are theoretical objects based on a mathematical concept, that, although useful for understanding the essence of fractality, are affected by numerous constraints when geographers are considering terrestrial spaces and then their morphology and morphogenesis.

These geographical phenomena belong to the self-affine class of fractals. The increase in the variety of self-affine forms and processes is never-ending, as scientists from all disciplines have been introducing new constraints in order to better comprehend the motion that shapes these fractal forms. This variety has heralded the deployment of multifractals, which are introduced in the next chapter.



## Chapter 3

# From the Fractal Dimension to Multifractal Spectrums

“What our contemporaries lack most is conceptual audacity”

Charles-Pierre Péguy, 1996

When geographers want to apply a fractal approach to a problem that is being dealt with, the calculation of a fractal dimension does not fully satisfy their requirements. They need answers to questions for which the fractal dimension is not able to bring about a response. For example, two urban forms may have similar fractal dimensions even though their textures may be quite different, so how are these differentiated? Similarly, a geomorphologist may want to compare the fractality of a whole relief with that of multiple transects. Another classic question is: a town or landscape possesses a global fractal dimension, but what is the dimension for each individual quarter or for the different scenes in the landscape?

Faced with questions that they have encountered having determined a global fractal dimension, researchers across all disciplines have proposed new techniques and methods. This chapter does not intend to answer all of the questions that may arise, but it nevertheless widens the palette of fractal techniques available to geographers. It concentrates on lacunarity and codimension, which are simple extensions of the fractal dimension, before tackling the corrections deemed necessary when the power law is not perfectly linear, which is the rule rather than

the exception. Section 3.3 is dedicated to fractal scanning techniques. Finally, multifractals generalize the global fractal dimension in a different way.

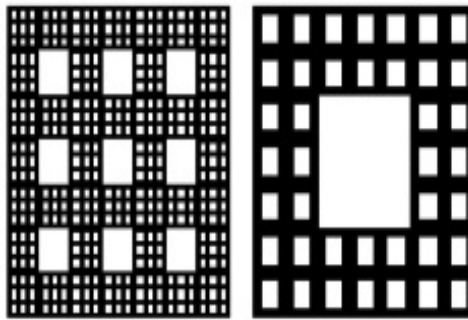
### 3.1. Two extensions of the fractal dimension: lacunarity and codimension

The fractal dimension is not very effective in terms of comparing textures. Fractalists have developed a new indicator: lacunarity. Then, again in order to produce comparisons between phenomena whose topological dimensions are distinct, such as between a line and a surface or between a line and a volume, meteorologists have invented another tool: codimension.

#### 3.1.1. *Some territorial textures differentiated by their lacunarity*

Briefly, the fractal approach was being used in order to define and classify image textures, notably in astronomy and medicine, as the fractal dimension was an indicator of coarseness. This method appeared to be better suited than the classic tools, such as the Haralick indicators deduced from the co-occurrence matrix, in distinguishing and cataloging irregular tissues.

These Haralick indicators measure multiple texture characteristics. For example, the homogeneity index reveals some uniformity, while entropy is an indicator of texture disorder.



**Figure 3.1.** *Two different textures with the same fractal dimension*  
(source: B. Mandelbrot [MAN 77])

Textures that are very different can have the same fractal dimension, however, as shown in Figure 3.1. The two forms have the same box counting fractal dimension although there is otherwise little similarity. The explanation for this is simple. The fractal dimension provides a good measurement of an object's irregularity, but it

does not take account of the spatial arrangement of its components. New indicators were therefore proposed within fractal geometry, particularly lacunarity, which was defined by B. Mandelbrot in 1983 (defined in [MAN 75]).

Lacunarity is a measure of an image's inhomogeneity. It evaluates the significance of holes in a binary object, which is represented in black on a white background. The larger the number of holes in terms of quantity, size and irregularity, the higher the lacunarity. Compared with the fractal dimension, which measures average irregularity, lacunarity equates to a variation coefficient. For all forms, high lacunarity is synonymous with low coherence, which is why this indicator will be a great help in the appreciation of the social coherence of a territory.

As with the fractal dimension, there are different calculation algorithms that are selected according to the data being handled. These algorithms are detailed in articles written by C. Allain and M. Cloitre [ALL 91] and the works of R.E. Plotnick *et al.* [PLO 93]. The creators of the *Fractalyse* and *ImageJ* programs have applied some of these algorithms, which enable the calculations of various types of lacunarity. We will revisit these in Chapter 4.

Above all, however, the calculation of lacunarity remains relevant for nonfractal objects [PLO 93], which is why various authors have verified its differential power in medical imaging and in remote detection. For example, S.W. Myint and N. Lam [MYI 05] have demonstrated that in an *Ikonos* image of the town of Norman, Oklahoma, the typology of urban textures is more precise when using lacunarity indicators rather than a simple fractal dimension or a spatial auto-correlation indicator. The values for lacunarity are very different for prairies and fallow lands, which allow a good understanding of different types of green spaces, which are very hard to differentiate using other indicators.

There is a default value for lacunarity; it is not scale invariant. It has different values when the scale is changed, which is why the variation in lacunarity as a function of scale contains more significant information relating to an object's texture than lacunarity that is evaluated for a single level. In spite of its good appreciation and discrimination of textures, lacunarity remains a virtually unknown tool in geography, a science that prides itself on analyzing territorial disparity and segregation.

### **3.1.2. Codimension as a relative fractal dimension**

The value of a fractal dimension is nearly always greater than the value of a topological dimension. It is therefore impossible to compare the irregularity of two

objects where their topological dimensions are not identical, such as a line and a surface. Fractal dimension is an indicator of the absolute and global irregularity of a phenomenon. It is sometimes necessary to use a relative indicator.

Codimension is that relative indicator, and is complementary to the fractal dimension. It is given by the formula:

$$Co = Dt - Df \quad [3.1]$$

where  $Dt$  is the topological dimension, in which the fractal (embedding dimension) is 2 if the fractal object is a line, and 3 if it is a surface.  $Df$  is the fractal dimension of the object being analyzed.

Whereas the fractal dimension is a gross value, codimension varies between 0 and 1. Forms for the most irregular chronicle series have a codimension of around 0; while the absence of irregularity is indicated by a codimension equal to 1. Interpretation of the codimension is therefore not very intuitive despite its usefulness. Indeed, the calculation of codimension enables a good comparison of fractal objects whose topological dimensions are different.

Consider a digital land model. It is possible to calculate its global fractal dimension, which will be greater than 2. For each transect of this digital land model, i.e. a line, the fractal dimension will be between 1 and 2. These two types of fractal dimensions are not homologous. Conversely, the codimensions of the digital land model and their transects are comparable, since they are all spread between 0 and 1. If the codimension of the transects is greater than that of the whole digital land model, the irregularity of the whole relief will be greater than that of its transects.

Clearly, this logic applies to any chart, a map indicating population density or land prices in a town. Meteorologists have proposed that this indicator make use of codimensions in order to compare fields of pressure, temperature and rainfall. However, like lacunarity, codimension is little valued in geographical studies in spite of its simplicity of evaluation.

Other indices are specific to the problems concerned. For example, in chemistry the combination of atoms into molecules takes place under certain linking conditions. These specific indices handle questions that are formalized by a network and are therefore subject to a graphical approach. Chemists have therefore established some fractal dimensions for spread and then connectivity.

There are very few applications using these specific dimensions, as described by J.-F. Gouyet [GOU 92], in geography. They do, however, appear to be well suited to the study of transport networks, as proven by the works of C. Genre-Grandpierre

[GEN 99] and N. François [FRA 97]. The spreading dimension is an indicator of accessibility to the nodes of a network and takes into account the connectivity of this network.

### **3.2. Some corrections to the power laws: semifractals, parabolic fractals and log-periodic distributions**

A second issue that needs to be addressed, which we have hardly raised in the previous chapters, is that if mathematical fractals are infinite, in reality however, fractal forms are limited to two extremes of considered levels. Town size or the temperature read by a weather station are never infinite. Similarly, the upper limit of a landslide depends on the size of the mountain environment that is reshaped by the slide.

These physical constraints are visually represented by a bend at both ends of the bi-logarithmic adjustment lines that are used to determine the slope, and then the fractal dimension. This is notably the case in a rank-size rule applied to towns in one region or state. All programmers and users of fractal analysis are well aware of this rule.

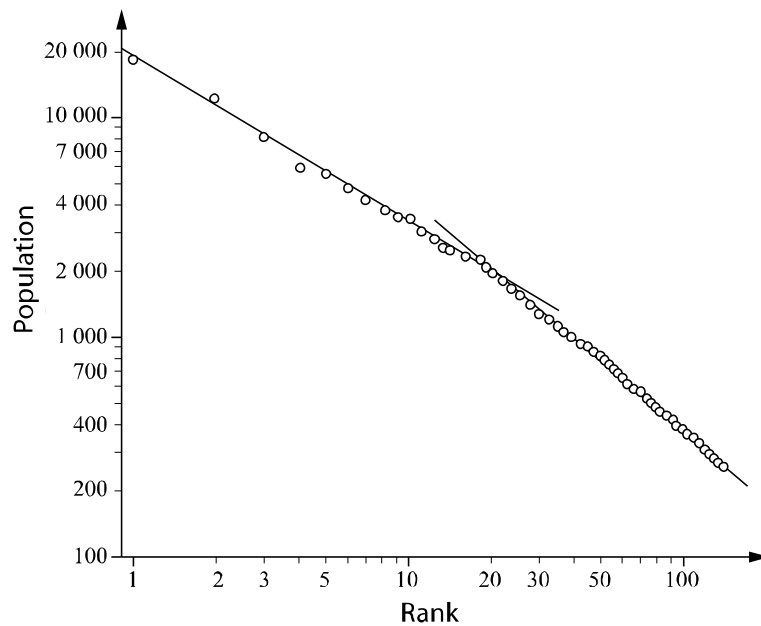
Sometimes, these two limits correspond to values for the resolution and spread of the phenomenon examined, but this is not mandatory; and beyond this well-known fact, most graphics produced from experimental data do not provide a perfectly linear straight line. In order to handle these constraints pertaining to observational data, at least three solutions are possible.

#### **3.2.1. *Semifractals and double or truncated Pareto distributions***

First, one or more crossovers appear, not just at the extremities but occasionally in the very body of the adjustment line. A reduced number of internal crossovers indicates that the object being studied should be described using two or three fractal dimensions. This model is a semifractal, a geographical phenomenon that will be characterized by two or three fractal dimensions.

Fractal dimensions are obtained by slicing the power law into two or three sections, then proceeding to log-log adjustments on the subsections identified. When slicing is only carried out on two sections, some authors prefer to use the term double or truncated Pareto distribution. This solution is illustrated in Figure 3.2, which depicts a rank-size rule of populations in communities from a rural county in France.

Identification of a semifractal denotes that fractality is generated using two or three different mechanisms. For example, a change in the fractal dimension within a vast river network will be attributed to a tectonic modification or to conflicting climatic conditions. These breaks between two fractal conditions should not be confused with a variation in the fractal dimension induced through a change in the scale of the object being studied. It is unusual for the fractal dimension of a single town to be equal to that of the same town and its suburbs.



**Figure 3.2.** *Semifractal of a rank-size distribution*

There are many examples of semifractals. P. Hubert [HUB 89] illustrated the semifractal phenomenon by studying rainfall patterns in the Sudan-Sahelian climate. More recently, I. Brissaud [BRI 04] has considered examples of semifractals in a study on alpine irrigation networks. He observed that the Briançon irrigation network was characterized by two fractal dimensions. He attributed this double fractality to the mountain terracing and to history. Conversely, around Cuneo a single fractal dimension equal to 1.52 was sufficient to characterize the irrigation network implanted into a virtually homogeneous relief zone. Finally, in Valais the mountain terracing and evolution of farming practices are at the heart of a network defined by three fractal dimensions.

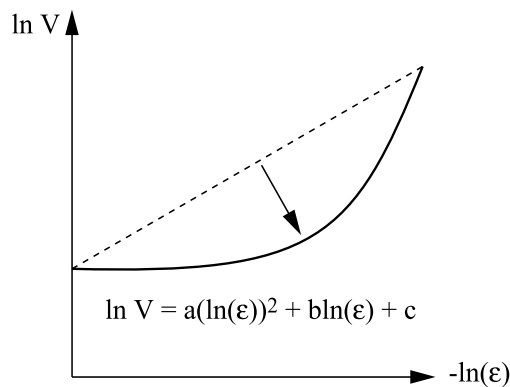
In climatology, a rupture between two fractal conditions is often induced by the synoptic duration, i.e. around 15 days. For a rainfall chronicle series, it is often necessary to calculate two fractal dimensions, above and below the level imposed by the rhythm of general atmospheric circulation conditions.

Sometimes it is difficult to make a distinction between a semifractal and a log-normal distribution, which also has a transition point. However, it is more a case of a transition point than a real rupture observed in a semifractal. Furthermore, for a log-normal distribution, this transition point is always localized close to the median. Proof by contradiction therefore indicates that a crossover, away from the median, is solid proof of the existence of a semifractal.

Finally, the observational data may lead us to consider more complex semifractals. Thus, Q. Cheng *et al.* [CHE 01a] showed that drainage basins of the Monts Oak moraine around Toronto are well described for the small outflows by a single fractal dimension, but the large basins are actually multifractals. A multifractal structure follows on from a fractal within the same network. This example shows that multiple transitions can be spotted between distinct fractal systems.

### 3.2.2. The parabolic fractal model

A second type of irregularity in the log-log straight line leads to a parabolic fractal model being favored. The parabolic fractal model, advocated by J. Laherrère [LAH 96] and spread into geography by the Avignon team, is a second type of generalization. This solution is used when the geographer cannot distinguish any clean slices segmenting the power law, but observes a continuous drift in fractality, and therefore a curve (see Figure 3.3).



**Figure 3.3.** *Parabolic fractal* (source: M. Forriez [FOR 07])

In this study, starting with urban populations from the United States, J. Laherrère showed that a linear power law, although strongly correlated with the data ( $r^2 = 0.994$ ), gave an extravagant extrapolation of the total US population in 1990 of close to 4.6 billion. Conversely, the extrapolation of a parabolic adjustment gives a population of only 237 million, which is closer to the demographic reality of that country. He therefore proposed to progress to an adjustment that was not linear, but polynomial.

A parabolic fractal minimizes very large events, such as the primatial effect of an urban hierarchy. In any event, J. Laherrère [LAH 98] detected that Nigeria's petroleum reserves were greatly reduced when a parabolic adjustment was favored over a linear fractal adjustment. Later, he showed that this same parabolic route applied to worldwide petroleum deposits, except for those in North America, which had initial reserves of over 50 million barrels. Identical reasoning was applied to forest fires and A. Mangiavillano produced a brilliant demonstration on this in his thesis [MAN 08].

As with the previous case concerning the choice of semifractal, when data are arranged according to a curve rather than a straight line, another solution is to choose another distribution, which is often log-normal. This is moving away from the domain of fractals, however, since only power laws express scale invariance.

Thus, in only considering towns, and therefore populations above 2,000 inhabitants, an adjustment using a power law is acceptable but, in classifying communities according to their population, the log-normal distribution, otherwise known as Gibrat's law, is a better match. Geographers are no longer considering the same set of data, however, and have suppressed the appropriate distinction between rural and town communities. We will go into this issue in depth in Chapter 5, which is devoted to the rank-size rules.

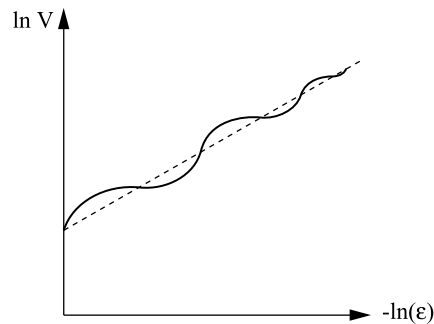
### **3.2.3. *Log-periodic distributions***

This is a third scenario that has recently been discovered: a number of phenomena follow a power law, coupled with a periodic evolution. Log-periodic distributions convey periodicity in scale changes [FOR 07], an acceleration or deceleration phenomenon. These distributions are therefore synonymous with hierarchical organization and fractality. Figure 3.4 illustrates this type of distribution, which is well known to physicists [NEW 95]. Periodicity is not unique to the power laws and may introduce oscillations in a parabolic fractal.

Log-periodic distributions have been the subject of much research, since they appear to better adjust the data from a great many risk situations, notably earth

tremors. D. Sornette has published several articles on the application of this formalism to earthquakes and financial crises and to their premises and their responses [SOR 98, SOR 00]. The pattern of aftershocks following an earthquake conforms well to this schema. The aftershocks reduce in intensity following the quake, over increasing time intervals. This aftershock deceleration follows on after a maximum or critical earthquake.

J. Chaline [CHA 09] has shown that this log-periodic distribution can be applied to the extinctions of major species while, in the same book, I. Brissaud applies it to the particle acceleration race, the history of jazz and the evolution of religious orders in the Middle Ages. More disturbingly, this formalism has detected an explosive crisis in the world's economy in around 2060 following the accelerated repetition of crises. P. Grou and L. Nottale devote several pages to this [CHA 09]. These few examples show the range of applications of this formalism.



**Figure 3.4.** *Log-periodic fractal*  
(source: M. Forriez [FOR 07])

We are confronted with two difficulties in the recognition of this distribution in geography. On one hand, the tracking of small irregularities before or after a crisis comes to pass [BRI 07]. This difficult tracking process for physical factors, such as an earthquake, becomes extremely arduous for social phenomena. On the other hand, each example quoted only deals with chronological series. The absence of the application of this distribution in two-dimensional space remains a void that young researchers are invited to fill.

In spite of everything, this distribution merits the attention of social science specialists, since it may be interpreted as a succession of increasingly rapid logistical evolutions. Yet these types of evolution seem to better reproduce mechanisms for growth – notably urban growth – than the continuous exponential evolution advanced by practitioners of urban geography, including those who attach significance to cycles of innovation for which generalized logistical diffusion has

nevertheless long been accepted. Exploration of this research path seems to us to be full of promise.

### 3.3. A routine technique in medical imaging: fractal scanning

Another delicate situation occurs when the global fractal dimension masks local fractal dimensions, which vary in time and space. Thus, in a town urban tissues do not offer an identical irregularity. The build up in certain quarters is more irregular than that of other urban areas. It is therefore practical to attribute a fractal dimension to each quarter, then to rank these quarters according to their degree of irregularity. Still in a town, it is logical to assume that the fractal dimension changes when we move out of the center, since the center of a town is denser and more volatile than the more ordered spaces of the 19<sup>th</sup> Century.

In medical image analysis, this process was generalized very early on. The image is sliced into subimages, and for each of these a fractal dimension is calculated. Specialists in medical imaging use the term *fractal scanning* to describe this set of techniques, which consist of attributing a specific fractal dimension to different parts of an image. We will therefore use this term in this book.

From this simple approach, many types of generalization are available, depending on the questions asked by geographers. First, in many computer programs the size of the subimages is adjustable, which enables changes in the fractal dimension to be analyzed based on the choice of subimage size. This approach, which has become routine in the examination of medical photographs – notably in order to detect and enhance different tumors – is as yet little used in geography.

This approach is, however, available in the *ImageJ*, *Fovea*, *ENVI*, *Idrisi* and *Landserf* software applications, and also, no doubt, in many others. C. W. Emerson *et al.* [EME 99] used fractal scanning in order to study variations in the fractal dimension of *normalized difference vegetation index* images. They varied not just the size of the windows, but also the pixel sizes and showed that the irregularity of agricultural spaces, which is appreciated by a fractal dimension, was increased by reducing the window size, while that of urban spaces remained identical for the town of Huntsville, Alabama.

Another generalization, which is related to fractal scanning, consists of processing a radial dilatation from a center, then calculating successive fractal dimensions. In geography, this type of approach, which is included in the *Fractalyse* software application, was implemented in order to analyze changes in the fractal dimension relative to a central point, or to set the limits between the town and the

countryside. M. L. de Keersmaecker [KEE 03] has given a convincing example of this approach relating to the agglomeration of Brussels.

A third approach compares the urban tissue from various quarters. P. Frankhauser [FRA 03] did this for several European metropolises, and D. Badariotti [BAD 05] applied it in convincing fashion to the town of Strasbourg-Kehl. In addition to the dense tissue of town centers, he distinguished various quarters of expansion. He then proposed two indicators, constructed from fractal dimensions, which enabled most of the tissue analyzed to be properly identified. Figure 3.5 shows this practice diagrammatically for three urban areas of the French Riviera (Menton, Nice and Grasse).



**Figure 3.5.** *Fractal scanning on three urban areas of the French Riviera*

This fractal scanning technique, which is well practised in medical imaging, should not be confused with a multifractal approach. It is still a monofractal approach, applied to binary spaces, and therefore to sets, in the mathematical sense of the term. In fractal scanning, the fractal dimensions vary as they are calculated on different parts of an object. The term multifractal defines a single object whose irregularity is described by multiple fractal dimensions. In concrete terms, each urban quarter possesses its own fractal dimension. It is unusual for town-center quarters to have the same fractal dimension as those in urban peripheries. This confusion between fractal and a multifractal scanning is generalized among geographers, as well as in some of our previous works.

Another area of confusion should be highlighted and discussed. Global fractal dimension, which is calculated on a set, changes if the initial resolution of this set is modified. For example, in the case of a town, preserving the same perimeter and based on two binary images – the first from a Landsat 5 photograph and the second provided by the Ikonos satellite – the results obtained are different. M. Forriez

[FOR 10] demonstrated this in his thesis based on the example of Avignon. Unequal fractal dimensions obtained by modifying the resolution are not comparable to those from fractal scanning, but they are not always multifractal functions. The vast majority of geographers still confuse these different situations and attribute the term multifractal to simple monofractals.

Indeed, during research, geographers are led to calculate successive fractal dimensions by modifying spatial and temporal limits or object resolution. They then make use of multiple continually global fractal dimensions, but these multiple dimensions are not comparable to a multifractal formalism.

Occasionally geographers may preserve all the information contained in a log-log straight line or curve, instead of resuming calculations using a simple fractal dimension determined from the slope of the straight line or curve. Some ecologists call this curve a fractogram. This technique is illustrated in the thesis written by A. Mangiavillano [MAN 08] and more recently in an article by I. Thomas *et al.* [THO 11], who showed sufficient interest in it to classify fractal urban spaces.

### **3.4. Multifractals used to describe all the irregularities of a set defined by measurement**

This first multifractal approach, which will be developed in Chapter 6, is in three parts. In section 3.4.1 we define a multifractal using a concrete example with no mathematical formalism. Then, in section 3.4.2 we present the two most classic multifractal functions: generalized dimension spectrums and the singularity spectrum. Finally, in section 3.4.3 we will open the window a little on possible generalizations.

#### **3.4.1. Definition and characteristics of a multifractal**

Although first used in studies on turbulence, we are introducing the multifractal concept based on the more geographical example of a town, so that we can obtain a better grasp of the difference between fractal scanning and the multifractal approach. When a town is represented by a binary image in black and white, with black representing built-up areas and white the open areas such as roads and parks, it is possible to calculate a global fractal dimension.

If the same town is represented in shades of gray, as in a satellite photograph, by proceeding through successive thresholds, multiple binary classifications of the same urban entity are possible. Thus, it is possible to distinguish towns with low-rise

buildings from others or, on the other hand, those with high-rise buildings. Other distinctions are quite clearly imaginable.

For each black and white image obtained through thresholding and then binarization, it is easy to calculate a fractal dimension. If this fractal dimension estimated for all binary configurations of the same object remains constant, then the town is a simple fractal, otherwise called a *monofractal* or *homogeneous fractal*. However, as a general rule, fractal dimensions are not identical, and instead of a single fractal dimension several are obtained, that form a function. The object studied, in this case a town, is then described as a multifractal.

Whereas fractal formalism is applied to a mathematical set, such as a Cantor set, *multifractal formalism* is applied to the measurement that describes this set, such as the measurement of rainfall for a region, population densities, numbers of businesses in a town, or even the numbers of museum visitors. Multifractal formalism enables responses to specific questions on densities and concentrations and not only on the presence or absence of a characteristic [BRO 10].

From this, multifractal formalism applies to fractal formalism measurements that, at least initially, principally concern sets. For example, it is easy to calculate a global fractal dimension of the localization of mineral deposits, notably diamonds. If we link this set of localizations to the price that is based on the size of the diamonds found in each deposit over all localizations, it is then associated with a measurement. This area of measurement – in this case that of potential prices – stems from multifractal formalism.

If we return to the initial schema of the construction of a fractal form produced through the iteration of a generator over a prior initiator, as described in Chapter 2, it is easy to provide a new interpretation of a multifractal. The multifractal form is a combination of at least two fractal generators, and this combination may be additive or multiplicative. B. Mandelbrot [MAN 05] has provided some examples of this in his works on the markets. These combinatorial cascades form the basis for increasingly complex spatial forms or temporal curves.

In order to better appreciate the interest in multifractal formalism, we must compare it to traditional statistical analyses that are based on examination of the standard deviation or variance, and of third and quart moments. These moments assess dispersion; the mean irregularity of data handled. Conversely, multifractal formalism measures all of the irregularities and discontinuities of a field or series. Multifractal formalism provides a representation of all irregularities, big and small, across all scales. Revealing the multifractal characteristic of a chronicle series or spatial field comes back to characterizing it at the time as being either multiscale or multi-intensity.

We can better understand the interest in this approach for geographers who are continually confronted with this double-sided problem.

More precisely, a multifractal function represents the large-scale propagation of irregularities. Thus, a discontinuity may be imposed at all levels, or conversely may only be identified at certain levels. On a map of France, municipal boundaries disappear, although they are still present on a county map. Multifractal formalism easily distinguishes these two situations and is now a favored tool for segmenting images and highlighting limits that fluctuate according to the levels of objects.

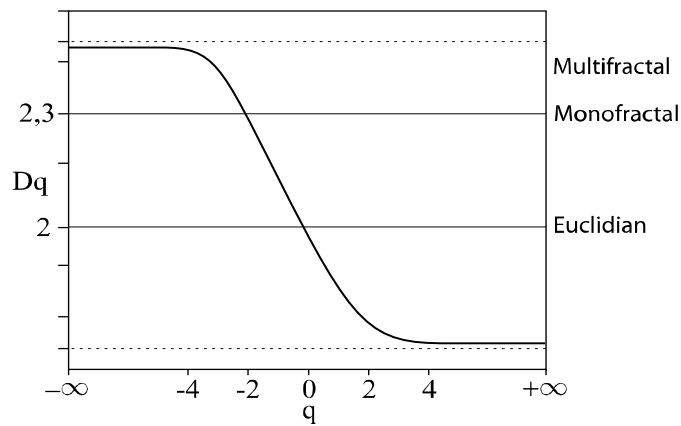
We can better understand the interest in multifractal formalism from the point of view of formalizing multiscale geographical discontinuities, differentiations and disparities. Without always following the underlying distribution of the data being studied, classical geographers focus their attention on specific cases and have created some brilliant monographs. Statistical geographers prefer to characterize this mean dispersion or irregularity through the value of the standard deviation or some other indicator of this type, such as entropy. Finally, multifractal formalism provides a complete representation of all deviations and their temporal localization, for a chronicle series, or their spatial localization for a satellite image or map.

#### **3.4.2. *Two functions to interpret: generalized dimension spectrum and singularity spectrum***

There is a particular abundance of scientific literature on multifractals, but it is highly technical [ABR 02]. Geographers must master complicated mathematical formalism, which no doubt explains the scarcity of geographical studies using this formalism, and occasionally the confusion that has historically arisen. In France, however, we presented the first example of the correct geographical application at the *Géopoint* 2010 and then *ThéoQuant* 2011 symposiums. Northern Mediterranean coastal ports were classified according to the irregularity of their urban tissue. This example will be developed in Chapter 6. Let us first try to summarize multiscale technical components in this preliminary approach as simply as possible.

Essentially, multifractality is analyzed based on two functions described as spectrums. The first of these, which is called the *generalized dimension spectrum* ( $Dq$ ) is represented in Figure 3.6. For a two-dimensional object, such as a town shown through a satellite image, dimensions that are all equal to 2 indicate the absence of fractality. The objects and this space are Euclidian. A constant value of these generalized dimensions, again illustrated by a straight line parallel to the x axis, but with a  $Dq$  value above 2, is representative of a monofractal. In this case, there is no variation in irregularity and fractality remains constant. Finally, a

multifractal is distinguished by a decreasing function, which very often takes on a logistical form.

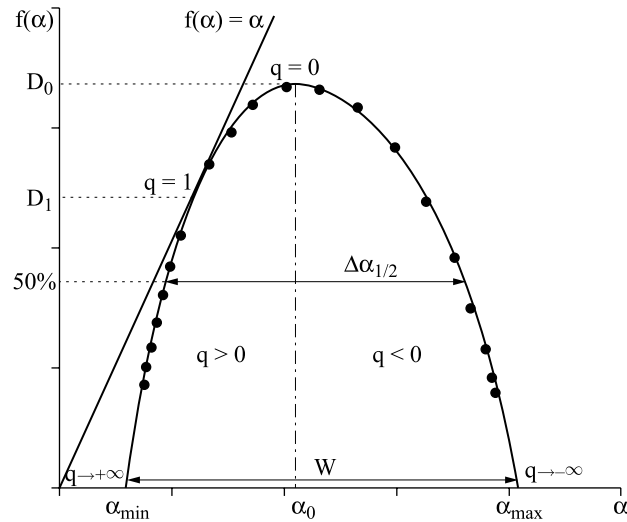


**Figure 3.6.** *The generalized dimension spectrum*

The slope of the  $Dq$  function provides some original information. A steep slope indicates a regrouping of highly irregular values, while a slight slope is synonymous with a more equal distribution of irregularities within temporal or spatial scales.

Figure 3.7 illustrates the second calculated spectrum, the *singularity spectrum*,  $f(\alpha)$ . For a monofractal, the values of  $f(\alpha)$  do not change, and the curve is reduced to a single point. Conversely, for a multifractal the curve takes on the shape of a bell. The maximum value, which is at the top of the bell, provides the fractal dimension of the base. By tracing the straight line from the origin at a tangent to the curve, we obtain the *global fractal dimension*, which is in fact the most likely. The curve itself indicates the two extreme limits of the variability of fractality. More generally, the spectrum indicates how many points of the signal – the chronological series or of an image – have a regularity equal to  $\alpha$ .

The examination of the singularity spectrum is more synthetic. The wider the curve; the greater the irregularity of the coarseness. Conversely, a very narrow curve indicates that the phenomenon studied is nearly monofractal. The maximum point corresponds to the Hurst coefficient, which is another indicator of irregularity. A value close to 1 indicates few irregularities, while a value of around 0 corresponds to strong irregularities.



**Figure 3.7.** *The singularity spectrum*

The virtually asymmetrical appearance of the singularity spectrum localizes these irregularities. A left-hand bulge indicates that irregularities strike before all of the little temporal or spatial structures. A right-hand bulge indicates that it is the large structures that are the most intermittent. A number of authors have even proposed that the coefficient of symmetry should be calculated in order to classify spectrums, and to draw conclusions from this in the distribution of irregularities on different scales. In this way, L. Telesca *et al.* [TEL 04] differentiated seismic sequences of the Frioul, Les Marches and Irpinia regions. Les Marches can easily be distinguished from the other two regions.

Finally, without presuming to be complete, most computer programs provide other aids to understanding, such as the exponent spectrum, also known as the division function, the scaling moments function and the codimension spectrum, which enable comparisons to be established. These programs are frequently used in meteorology and astronomy and we will discuss these again in Chapter 6.

### **3.4.3. An approach that is classical in geosciences but exceptional in social sciences**

Geoscience researchers were very quickly taken with multifractal formalism. On a global scale, J. S. Gagnon *et al.* [GAG 06] showed that Brownian and Lévy motion monofractal models reproduced topographical complexity very poorly. Multifractal

approaches, however, simulated topographies in accordance with the observations of the *ETOP05* or *GTOPO30* digital terrain models.

In meteorology and hydrology, multifractal analyses are now *de facto* analysis tools. They are applied to intensity-duration-frequency rainfall curves, pluviometric fields and river flows. From spectrums, hydrologists [MAJ 04] have deduced universal parameters that have helped them to establish river classifications so as to better regulate flooding. Finally, we highlight the recent studies dedicated to sun spots, which agree with long-term correlations. Solar activity is therefore not stationary and has had an impact, however poorly appreciated, on climate change over the past 30 years.

Studies calling on multifractal treatment are also commonplace in seismology. Thus, A. Zamani and M. Agh-Atabai [ZAM 09] have analyzed the temporal organization of earthquakes in the Alborz and Zagros mountain ranges in Iran. They have applied multifractal formalism to the time intervals between successive earthquakes. The decrease in the  $D_q$  function, which is very slow for the Zagros region, suggests a dispersed distribution of the times of earthquakes. Conversely, for the Alborz region, the slope of the  $D_q$  spectrum is steep. Therefore, earthquakes here happen in bursts, with periods of strong activity followed by periods of calm.

While multifractal formalism is less widely used in social sciences, it has still been shown to be very relevant. For example, De Wang *et al.* [WAN 10] compared the territorial irregularities of different uses of land on the Loess Plateau, deep in the heart of China. The forms of habitat and water spaces were the least irregular. The curve, which was practically horizontal, was synonymous with monofractality. Forests and green spaces were multifractals, which seems *a priori* to be a contradictory set of results that might be explained by a generalized population distribution and therefore the regularity of its irregularities.

This approach is applicable to images or maps, to chronological series and also to series representative of displacement. Thus, F. Schmitt and L. Seuront [SCH 02] applied this formalism to the displacements of the crustaceous copepods of zooplankton. Their conclusion was an abnormal diffusion. This diffusion, which differed from the Brownian and Lévy models, tended to optimize the volume visited for a given energy expenditure.

The application of similar treatments to human displacement would be advisable, since using this approach X. Li and P. Shang [LI 07] classified the Beijing road traffic. Traffic generated by the internet is increasingly serving to test the new multifractal methods placed by statisticians at the disposal of the research community, and therefore geographers.

#### 3.4.4. *Three potential generalizations*

Multifractal formalism, which extends the simple fractal approach, is itself subject to generalization. A first generalization, which is the simplest, consists of deducing indicators from these curves that define the scale of variation in fractality. Based on the generalized dimension spectrum, the difference between  $D_{-5}$  and  $D_{+5}$  can be calculated, which is an indicator of the complexity of the scale. The greater the difference, the more irregular the object considered will be in multifractal terms. As multifractality is in part synonymous with scale complexity, it is becoming easier to classify pluviometric series, towns and any other geographical phenomenon in terms of scale complexity. Bringing about a little numerical rigor and precision is not a luxury, as both are commonplace in relevant complexity and multiscale incantations in the writings of modern-day geographers.

The values  $D_0$  (capacity fractal dimension),  $D_1$  (entropic information dimension) and  $D_2$  (correlation dimension) lead to other interpretations.  $D_2$  is equivalent to Simpson's diversity index, which is prized by ecologists. Some authors [TAR 06] prefer to make use of the difference between the capacity fractal dimension and the entropic information dimension:  $D_0 - D_1$ . The bigger the difference, the more significant the irregularities are.

Finally, on the same subject we should mention the universal multifractal approach that is widely used by hydrologists. From the many spectrums, two or three parameters are derived that are used by hydrologists to classify pluviometric and hydrological conditions. Codimension  $C_1$  measures inhomogeneity, which is the mean multifractality. This is equal to 0 for a homogeneous field. The multifractality index, denoted by the symbol  $\alpha$ , indicates the degree of multifractality in the chronicle series or field observed. It is therefore possible to classify phenomena and to better understand their irregularity – notably that of extreme values.

A second generalization has been inspired by normal statistical treatment. Using two sets of values, it is possible to progress to a comparison – either empirically or by calculating a simple or even multiple correlation or regression – when multiple explanatory variables are used. *A priori*, there is nothing to stop us proceeding in an identical fashion with singularity spectrums in order to calculate correlations that link them. This comes back to the analysis of the relationship between two phenomena based on their multiscale irregularity.

The implementations of such approaches remain exceptional. Let us consider, however, the study undertaken by B. Eghball *et al.* [EGH 03]. They compared irregularities in maize crops and in nitrate levels in soil during an experiment conducted in Nebraska from 1995 to 1997, which showed that a 25% reduction in fertilizer did not lead to significant maize crop reductions. In a similar vein,

A.N. Kravchenko *et al.* [KRA 00] established a genuine correlation between agricultural yields and the slope of terrain.

There is a third generalization, which is often quoted in studies into turbulence: divisible and indivisible cascades. This formalism, which is well explained in the thesis by P. Chesnais [CHE 01b], does not seem to have inspired the least bit of geographical research. Physicists at Lyon's ENS have, however, applied this extension of the multifractal approach to the study of internet traffic.

### 3.5. Conclusion

Spatiotemporal phenomena, which are represented by sets described according to a measurement that is commonplace in geography, are more relevant to a multifractal approach than to a fractal technique. Whereas fractal formalism contents itself with global dimension calculation, multifractal methods describe and localize singularities and irregularities at all levels. They should be considered as soon as the geographer studies a field described according to a measurement, whether it be physical, such as altitude, or human, such as population density or any other social characteristic.

Nevertheless, before resorting to this complex formalism, it is sometimes useful to call upon other fractal indicators. With the exception of different forms of fractal scanning, which are widely used in urban geography, the other indicators are still all too often neglected.



## Chapter 4

# Calculation and Interpretation of Fractal Dimensions

“Even a function which is impossible to design or imagine  
can become a perfectly respectable Fourier series”

Barbara Burke Hubbard, 1995

Chapters 4, 5 and 6 outline a number of algorithms for calculating fractal dimensions and multifractal functions. Readers who only want to understand the fractal approach, without applying it to their own observational data, may be discouraged by the technical content of these chapters. In this case they are not bound to read them. For the more involved reader who is keen to benefit from these new tools, which are well suited to the analysis of irregularities in the land, these chapters mention some pitfalls to be avoided. They cover the algorithms that are expressed as computer programs in the Appendices and that are implemented in these chapters in depth.

Chapter 4 comprises four parts. Section 4.1 contains a brief presentation of the data that will be handled by the algorithms discussed. Section 4.2 sets out successive tests that a geographer should perform in order to identify the type of fractal that he or she is looking for, whether this is in a simple form, such as an urban sprawl, or a complex process illustrated by a highly irregular chronological sequence. Then, in sections 4.3 and 4.4, some algorithms for calculating the fractal dimension are

formulated. As even a large book would not be sufficient to detail them all, our choice has been guided by questions from geographers. Some of these algorithms will be translated into small computer programs, written in *Mathematica* language and shown in the Appendices.

Chapters 5 and 6 are structured using the same principles. Chapter 5 deals with fractal frequency distributions, principally from the modelization of the rank-size rules; whereas Chapter 6 is devoted to multifractal formalism.

#### 4.1. Test data representing three categories of fractals: black and white maps, grayscale Landsat images and pluviometric chronicle series

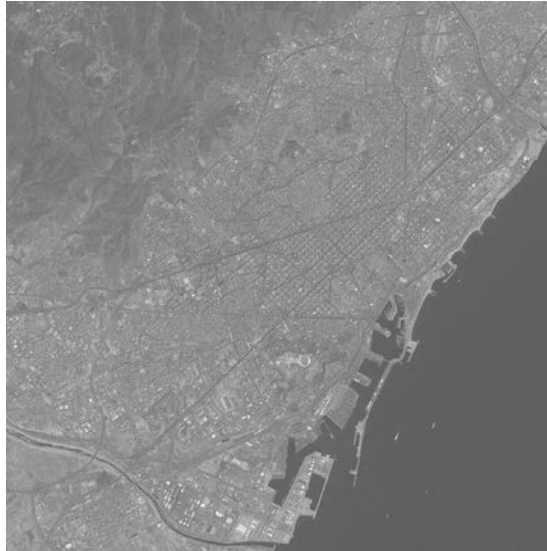
In order to demonstrate the use of the fractal approach, we apply three groups of data to techniques presented in this chapter. First, in order to enable comparative studies, we have constructed a set of three maps from *Mathematica* databases. Each map represents a view of the World according to the inhabitants of three major cities: Moscow, Paris and New York. These maps reproduce the distinction between the oceanic expanse and the continents in binary form. They are therefore comparable with urban maps, which distinguish buildings from open spaces. Figure 4.1 illustrates two of these representations, as seen by the residents of Moscow and New York. This example shows that the fractal approach can be generalized for all geographical data, whether they relate to physical or urban geography or a study of the perceptions or representations.



**Figure 4.1.** *The world as viewed by residents of Moscow and New York*

The second set of data comprises Landsat 7 images of six major ports along the north Mediterranean coastline: Barcelona, Marseille, Genoa, Naples, Athens and Beirut. Each agglomeration is a matrix 2,048 pixels squared. Figure 4.2 represents a

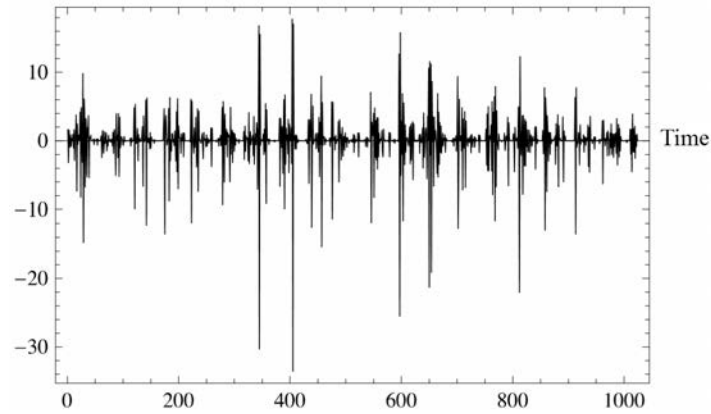
small area of Barcelona. Unlike the previous binary maps, these Landsat images hide more complete information, described in 256 shades of gray, each pixel corresponding to a measure or intensity.



**Figure 4.2.** *Part of the Landsat 7 image of Barcelona*

As the fractal approach is applied to the chronological, descriptive or representative series of a process, we have chosen decadal and monthly rainfall in a number of Mediterranean resorts, particularly Barcelona, Marignane and Beirut. These bring to light a third case, with quantitative values, which geographers are used to making use of in research. Decadal rainfall conforms more to the synoptic rhythm of types of atmospheric circulation than monthly rainfall, and its resolution is more detailed. These rainfall patterns are also interesting from the point of view of being arranged along a time axis. It will be easy to remove this temporal organization in the next chapter by classifying them as frequencies of recorded heights. Figure 4.3 illustrates the rainfall in Barcelona from 1,024 decades after a first-order differentiation, which is necessary in order to remove seasonal periodicity.

These three data applications are certainly not the only experimental situations that geographers encounter. For example, there is no array illustrated by a graph. However, this dataset is sufficiently broad for testing the pros and cons of the fractal approach.



**Figure 4.3.** *Differentiated decadal rainfall in Barcelona*

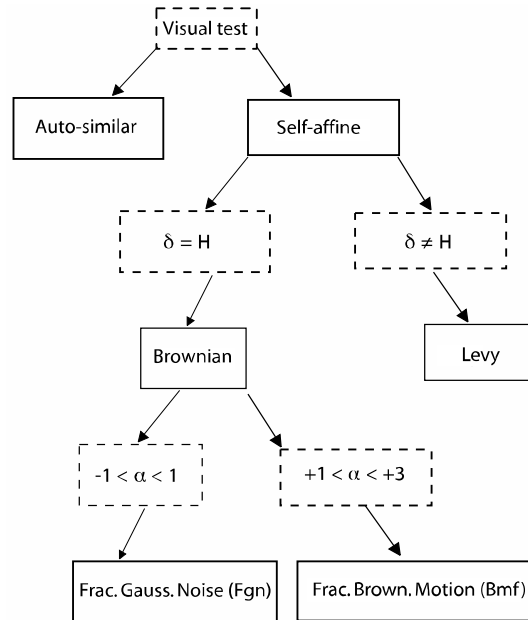
#### **4.2. A first incontrovertible stage: determination of the fractal class of the geographical phenomenon studied**

Whatever the type of data analyzed by a geographer, he or she must first of all identify the type of fractal that it represents. In order to do this, he or she carries out a series of tests, as mapped out by Figure 4.4, that are inspired by L. Seuront's book [SEU 10] and the works of N. Scafetta [SCA 10]. This diagram arranges the successive stages to be overcome for all fractal dimension calculations.

An initial visual test enables auto-similar fractals to be differentiated from self-affine fractals. An auto-similar fractal is identifiable from the identical repetition of the same pattern. This visual examination is sufficient to identify this repetitive, but exceptional, regularity as soon as we abandon the field of mathematics. If experimental data are the expression of an auto-similar fractal, it is possible to directly calculate a fractal dimension using one of the algorithms, which are detailed in section 4.3.1 of this chapter.

Since the geographical object is a self-affine fractal, it is usually necessary to carry out a second test in order to differentiate Lévy models from Brownian models. A third test, within the class of Brownian models, enables fractional Brownian motion to be distinguished from fairly simple fractional Gaussian noise.

These tests are imperative, as the calculation of the fractal dimension depends on them. Specifically, R. Voss [VOS 88] had very quickly learnt that the box-counting technique should not be used in the calculation of the fractal dimension of a self-affine object. Later, D.C. Caccia *et al.* [CAC 97] emphasized that not all techniques were appropriate for the calculation of the dimension of a self-affine fractal.



**Figure 4.4.** *Successive tests for determining fractal type*

Indeed, the dimension of a self-affine fractal is always calculated using a unique formula, which includes the value of  $H$ , but this Hurst coefficient is determined using a formula that changes the type of self-affine fractal identified. The choice of a good equation for calculating  $H$  is often imposed by the value of the power spectrum slope  $\alpha$  obtained through Fourier decomposition or wavelet decomposition. In the absence of this prior identification, some major mistakes have put a blemish on calculations of the fractal dimension.

#### 4.2.1. *Successive tests using Fourier or wavelet decompositions*

There is a subtle distinction between various self-affine fractal categories, which are the most common in the “real” world [DEL 06]. We must first of all differentiate between two main categories, namely the Lévy and Brownian models. In order to progress to this initial differentiation, two parameters need to be calculated: the Hurst coefficient,  $H$ , and a scaling exponent,  $\delta$ , which is determined through diffusion entropy analysis of the probability distribution of the series being studied [SCA 10].

A number of algorithms are used to determine the Hurst coefficient, such as the original R/S (rescaled range analysis and Hurst coefficient) method, which was developed by the hydrologist Hurst himself, the Fourier decomposition or the wavelet decomposition. Sometimes in ecology some authors have favored the adjustment of one part of the variogram or of the autocorrelation function, since significant autocorrelations in a chronicle series are a first indicator of fractal structure; and this list is not complete. There are advantages and disadvantages to every technique. For example, Hurst's R/S analysis is imprecise for series of less than 500 records.

Conversely, in order to define the scaling exponent, the approach suggested by N. Scafetta remains the most relevant. Defined by diffusion entropy analysis and programmed in C++, details of this can be found in his latest publication [SCA 10].

When the two parameters  $H$  and  $\delta$  are the same, the form or process examined falls within Brownian models. A difference between these two values reveals that the fractal phenomenon considered belongs to the class of Lévy models. In this case, the value of  $H$  to be used later to calculate the fractal dimension is taken from the value of  $\delta$  from:

$$\delta = 1 / (3 - 2H) \quad [4.1]$$

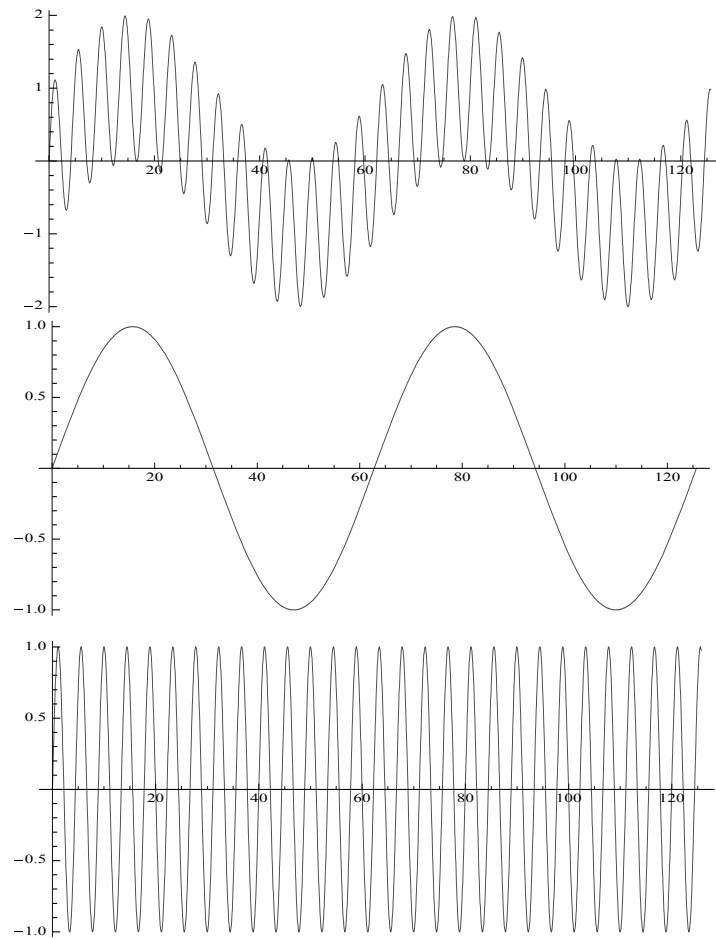
For a Lévy model, the value of  $H$  is therefore directly calculated. In his book, N. Scafetta applied this test to various series [SCA 10]. He thus showed that sun spots are arranged in time according to a Lévy flight. Furthermore, having highlighted coherence between the two fractalities of solar and atmospheric turbulence, he estimated that global warming should stabilize itself up until 2030–2040. This example demonstrates that these tests do not have a simply theoretical objective; they may have serious practical consequences.

Having established this first distinction, when the data represent a Brownian rather than Lévy model, geographers must then distinguish the different Brownian models. In order to achieve this, other tests are applied to produce a new demarcation between a fractional Brownian motion and a fractional Gaussian noise model.

Again, geographers have a number of tools at their disposal, in particular spectral analysis, wavelet decomposition and trended or detrended fluctuation analysis. Yet, we have emphasized above that spectral analysis or wavelet decomposition can also be used to distinguish a Lévy model from a Brownian one. Based on this fact, these two techniques are frequently used and merit a detailed explanation.

#### 4.2.1.1. Spectral analysis and determination of the scaling parameter

Spectral analysis or Fourier decomposition dates back to the 19<sup>th</sup> Century. Whether referring to a data set, such as the annual pass rate for the Baccalaureat, monthly attendance figures for museums, or decadal rainfall recorded in Barcelona, the curve corresponding to this data may be represented by a sum of elementary sine and cosine functions.

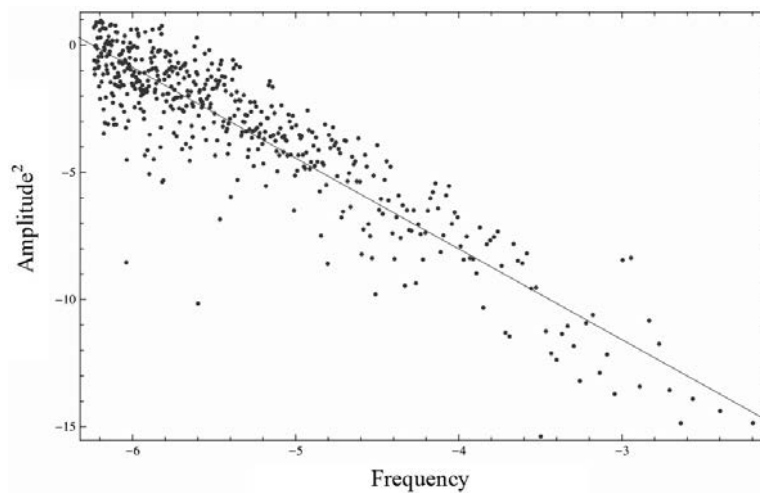


**Figure 4.5.** *Fourier decomposition*

Figure 4.5 shows how a theoretical series, A, is decomposed into two sine functions. Function B represents the global cyclical effect, while function C illustrates the details of curve A. The sum of these two elementary functions, B and

C, is immediately a good reproduction of function A. For data that are experimental but not theoretical, the Fourier decomposition generates not just two, but a large number of elementary sine functions.

Each elementary sine function corresponds to a frequency that is defined by both an amplitude and phase coefficient. By squaring the amplitude coefficients of every frequency, we obtain a new series: the power spectrum. The values of the power spectrum, which is sometimes described as the energy or variance spectrum, are then transposed onto a log-log graph, with the x-axis representing the frequency logarithm and the y-axis representing the logarithm of the amplitude squared (see Figure 4.6).



**Figure 4.6.** *The power spectrum of a Fourier decomposition*

When the points on this graph are arranged according to a sloping straight line, then the phenomenon studied is fractal. The slope of this line,  $\alpha$ , is determined by regression. This  $\alpha$  value is used as a test in order to decide between types of fractals within the wide range of Brownian motions. If the value of the slope is between -1 and +1, then the series is stationary [SER 10]. The data create fractional Gaussian noise. This series is said to be anti-persistent when the value of  $\alpha$  is negative, and is said to have low persistence for a value between 0 and 1. Furthermore, experimental data are the fruit of a simple Brownian process when  $\alpha$  is equal to 0. When the value of the slope is between +1 and +3, however, the variance of this series increases with the number of observations. It is not stationary. Finally, when the value of the slope is less than -1 or greater than +3, then some preliminary treatment is recommended that consists of integrating or alternatively differentiating

the original series. This situation, which is exceptional in geography, can be bypassed by proceeding to wavelet decomposition or by transforming the original series.

This procedure, which is described for a chronological series, is applied to an image or a volume and is enough for progression to a Fourier decomposition in two or three dimensions. Programming this is relatively straightforward.

Some preliminary treatment, which is explained in specialized works, is necessary in order to obtain regular spectrums before deducing the slope,  $\alpha$ , from this. The explanation of these balancing techniques is outside the context of this book, but we take them into account in the *Mathematica* programs provided in the Appendices. For example, the highest frequencies are excluded before proceeding to adjust the spectrum.

The Fourier decomposition suffers from some major defects. Although it is very effective when dealing with the analysis of gradient spaces, represented by a variable whose value changes gradually and regularly, this tool is blind in the face of severe discontinuities, clear limits and those which define geographical spaces, such as a break in the slope on a hill or the effect of the 1914–18 war on the French demographic curve. In fact, in order to describe a clear limit, it is necessary to mobilize all frequencies and therefore lose the benefits of Fourier decomposition. The spectrum obtained no longer enables the different levels to be well observed.

Furthermore, geographers should successively and separately study the area of data and frequencies. They are not able to look at both at the same time, which hampers any multiscale approach.

Finally, Fourier analysis, which includes elementary waves, is only useful for linear processes. The Fourier spectrum does not take into account correlations between levels, the significance of which we highlighted in Chapter 1.

#### 4.2.1.2. *Wavelet decomposition and determination of the scaling parameter*

In order to overcome these constraints, it is better to use wavelet decomposition, which was discovered by J. Morlet, an engineer who worked on handling data brought about through probing for layers of terrain that were potentially rich in petroleum. By applying Fourier decomposition, he was unable to isolate free discontinuities separating the sedimentary layers. He therefore proposed another form of decomposition, called wavelet decomposition, which was the starting point for numerous studies that applied the principle of Fourier decomposition.

As wavelet decomposition has become the incontrovertible tool for tackling the modelization of multiscale observations, we recommend a number of books on this

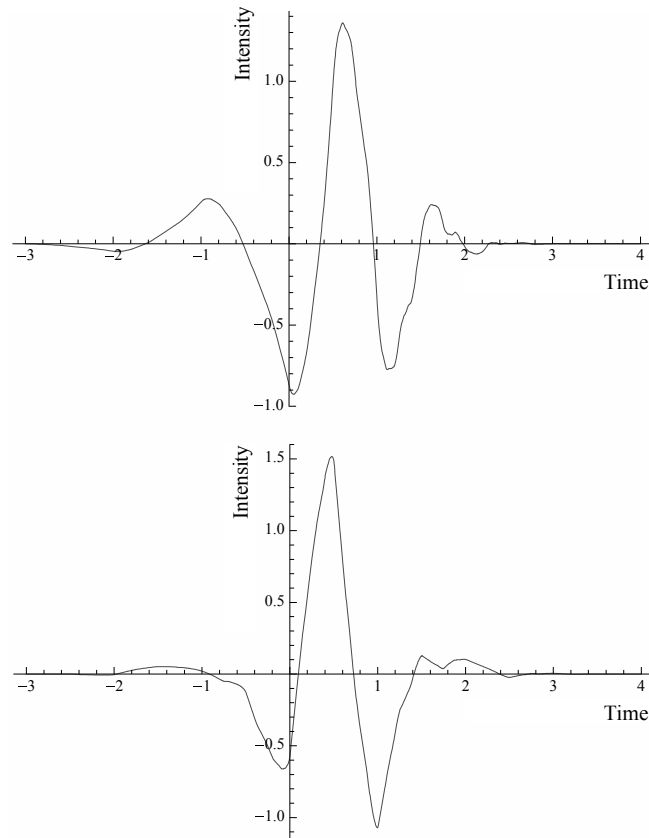
subject. By way of introduction, B.B. Hubbard's small publication is easy reading. Staying with the French language, this initiation is completed by the books written by P. Abry [ABR 97], [ABR 02], and A. Arneodo *et al.* [ARN 95], which are both very easy going. In order to be thrown into wavelet formalism, fractals and multifractals, the two volumes written by P.S. Addison and R. Gonzales [GON 08] highly informative texts [ADD 02] are essential.

Wavelet decomposition is original in two respects relative to Fourier decomposition. On one hand, sine and cosine functions, which are periodic, are replaced by a mother function, which is quite regular. This is chosen by the researcher based on the nature of his or her data. Many physicists and mathematicians have proposed a wide mother wavelet range, which often bear their names. Daubechies wavelets are the most widely quoted and used, but the *Mathematica Wavelet Explorer* software package offers many other solutions. This package is now included in version 8 of *Mathematica*, and is complemented by other very useful functions for the effective visualization and application of the results. This integration makes programming easier.

Of all the wavelets, several studies have suggested that the best results are obtained using discrete wavelets. Continuous wavelets provide skewed results, particularly when the Hurst coefficient,  $H$ , is greater than 0.5 [SCI 07]. Among discrete wavelets, the Daubechies and Symlets wavelets are the best performing. Figure 4.7 gives an illustration of this.

The mother wavelet is horizontally and vertically deformed through the effect of the signal. The wavelet coefficients, which are equivalent to the Fourier decomposition amplitudes, are thus a measure of the correlation between the signal shape and the contour of the deformed mother wavelet that follows the signal. As wavelets are constructed over a scale change operator, we can better understand their effectiveness using a fractal analysis, and more generally using any multiscale approach. They should become an incontrovertible tool across the whole of scientific geography, with its interest in multiscalar and multiscale phenomena.

More specifically, the signal – whether it is a chronicle series or an image – is decomposed into one general part and some details representing some irregularities. The general part is then itself decomposed into a new general part and more details, and so on. This, you may recall, is an iteration model similar to that of the construction of a fractal. This connection explains the effectiveness of wavelets in the study of fractals.

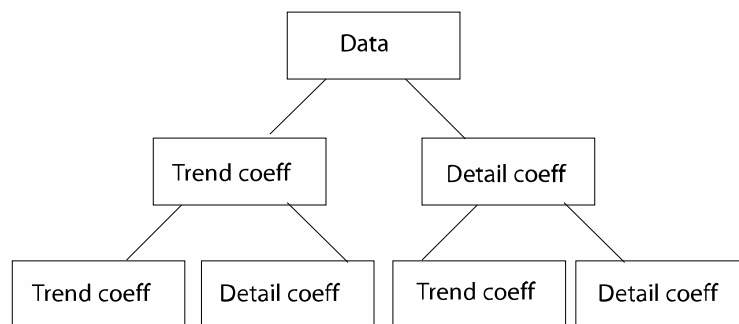


**Figure 4.7.** *Two mother wavelets: Daubechies and Symlets*

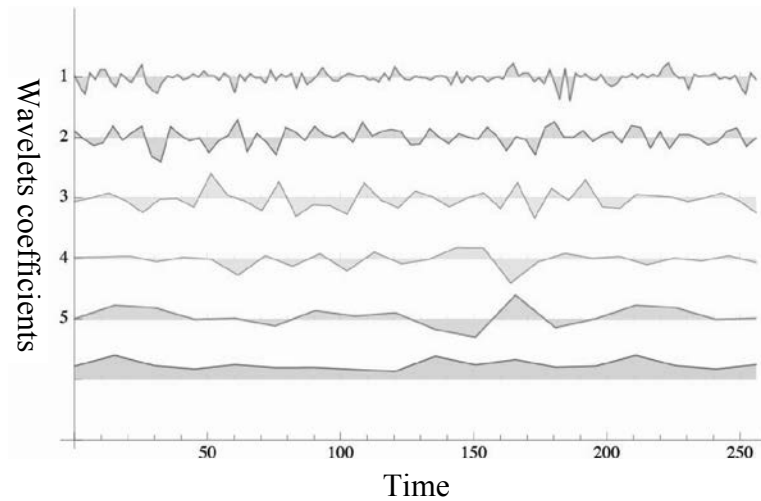
Figure 4.8 illustrates this decomposition, which enables the identification of signal irregularities or singularities on different scales. At each level, the intensity of these irregularities is effectively inscribed in the details. The wavelet coefficients, which are equivalent to the amplitudes and phases of a Fourier decomposition, therefore enable the localization of discontinuities and the simultaneous precision of their amplitudes in time or space.

This mathematical formalism offers several advantages [ABR 97, ARN 95]. First, with a Fourier decomposition geographers have two pieces of information on the geographical phenomenon being studied at their disposal – one in time and another in frequency for a chronological series, in terms of space and spatial frequency (or number of waves per image), respectively. However, these two pieces

of information are separate. Wavelet decomposition takes advantage of the two pieces of information simultaneously. It sees the wood and the trees at the same time. Wavelet decomposition resembles a multiscale microscope, as it is very well suited to the identification of various levels of geographical phenomena.



**Figure 4.8.** *Principle of wavelet decomposition*



**Figure 4.9.** *Wavelet transformation diagram  
(Barcelona monthly rainfall)*

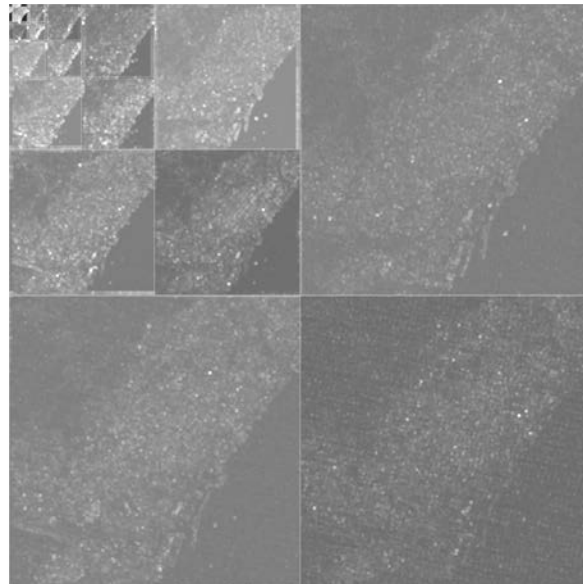
Figure 4.9 brings this multiscale decomposition principle to light. It is a diagram showing the wavelet transformation of a series of 256 monthly rainfalls in Barcelona. The bottom section of the diagram illustrates general trends, while the top section represents large-scale irregularities.

There are three points to note:

- First, the absence of a regular trend, with periods of dryness and higher humidity.
- Second, we observe a fairly generalized discontinuity, between months 170 and 190, indicating a cycle of around 15 years.
- Finally, large-scale but not generalized discontinuities are noticeable at around 45 months, that is at three years and nine months.

In real research, it would be useful to analyze some longer series, to compare multiple resorts and to perform some preliminary investigation in order to remove the seasonal cycle.

These comparisons can be made with other graphical schemas, such as a scalogram, which displays the squared value of the wavelet coefficients, i.e. their energy.



**Figure 4.10.** *Wavelet decomposition diagram of Barcelona from satellite imaging*

Furthermore, as with Fourier analysis, wavelet decomposition can be applied in two dimensions, and therefore for maps or grayscale satellite images. Figure 4.10 shows a wavelet decomposition of Barcelona on five levels.

Then, by squaring the wavelet coefficient values, an energy spectrum is produced that is equivalent to the Fourier decomposition power spectrum. Estimating its slope enables the fractal type corresponding to the observational data to be determined by adopting the rules already described for proceeding to a Fourier decomposition. The only difference is that the upper and lower limits widen the valid range of the test.

Effectively, while some bias is introduced with a Fourier decomposition, when the slope,  $\alpha$ , is below -1 and above +3 for a wavelet decomposition, there are values for  $\alpha$  below -3 and above +5 that indicate the presence of bias in the calculations. A value of  $\alpha$  between -3 and -1 reveals that the process is of the differentiated fractional Gaussian noise-type; whereas a value between +3 and +5 is synonymous with integrated fractional Brownian motion. In total, it is easy to distinguish five subcategories of Brownian model.

This first subtlety enables us to understand why a number of practitioners favor wavelet decomposition over Fourier decomposition. It widens the field of identification of fractal categories within the class of Brownian motions (see Table 4.1).

Additionally, various solutions enable the value of  $\alpha$ , and indirectly therefore  $H$ , to be determined. Some authors perform an adjustment on the means of wavelet coefficients for each level of decomposition, while others perform this operation on the standard deviation, variance or energy of these coefficients [ADD 02]. The value of  $H$  is dependent on this preliminary choice. For example, for an adjustment of the standard deviations of wavelet coefficients, the equation used to obtain  $H$  from the slope is expressed as:

$$\alpha = H + 1/2 \quad [4.2]$$

In conclusion, the best approach consists of producing a Fourier or wavelet decomposition, then calculating a power or energy spectrum and deducing from these the slope  $\alpha$ . From the values of  $\alpha$  obtained, it is possible to determine the type of self-affinity and therefore choose the best formula to apply in order to calculate the Hurst coefficient,  $H$ , which will be used to calculate the fractal dimension.

The value of the Hurst coefficient,  $H$ , is given by the relationship:

$$H = (\alpha + 1) / 2 \quad [4.3a]$$

for fractional Gaussian noise:

$$H = (\alpha - 1) / 2 \quad [4.4a]$$

for a fractional Brownian process:

$$H = (\alpha + 3) / 2 \quad [4.3b]$$

for a differentiated fractional Gaussian noise:

$$H = (\alpha - 3) / 2 \quad [4.4b]$$

for an integrated fractional Brownian process.

Table 4.1 summarizes the conclusions to be drawn from a wavelet decomposition:

| Slope = $\alpha$     | Fractal type                             | Value of H             |
|----------------------|------------------------------------------|------------------------|
| $\alpha < -3$        | Bias                                     | –                      |
| $> -3 \ \alpha < -1$ | Differentiated fractional Gaussian noise | $H = (\alpha + 3) / 2$ |
| $> -1 \ \alpha < +1$ | Fractional Gaussian noise                | $H = (\alpha + 1) / 2$ |
| $\alpha = 0$         | Brownian motion                          | $H = 0.5$              |
| $> +1 \ \alpha < +3$ | Fractional Brownian motion               | $H = (\alpha - 1) / 2$ |
| $> +3 \ \alpha < +5$ | Integrated fractional Brownian motion    | $H = (\alpha - 3) / 2$ |
| $\alpha > +5$        | Bias                                     | -                      |

**Table 4.1.** *Summary of tests following wavelet decomposition*

Examination of these equations and this summary table throws light on how critical these tests are, since the fractal dimension of all these phenomena is eventually calculated based on the Hurst coefficient. Without this preliminary step, it is possible to obtain fanciful values for the fractal dimension, for example values below the topological dimension, an error that has, however, been made in articles submitted to prestigious geographical publications.

#### **4.2.2. Decadal rainfall in Barcelona and Beirut are fractional Gaussian noise**

The algorithms described above have been applied to a rainfall series recorded for Barcelona over 1,024 years, and then to the same series measure for Beirut. In order to remove the seasonal effect, which is definite, and a possible trend to do with

dryness, a differentiation of 1 has been applied to the two raw data sets. However, we have also applied the tests to raw data and over a series of 256 months of rainfall.

Visual examination of the graphs is sufficient to show that these pluviometric series are not auto-similar but only self-affine, which is normal as the two axes – rainfall and time – are not of the same type. For Barcelona, the coefficients  $\delta$  and  $H$  are virtually the same (0.26 and 0.24 for the differentiated series and 0.39 and 0.37 for the raw series). These series are therefore indicative of a Brownian model. The value of the slope of the Fourier spectrum, which is between -1 and +1, indicates that the rainfall corresponds to a fractional Gaussian noise.

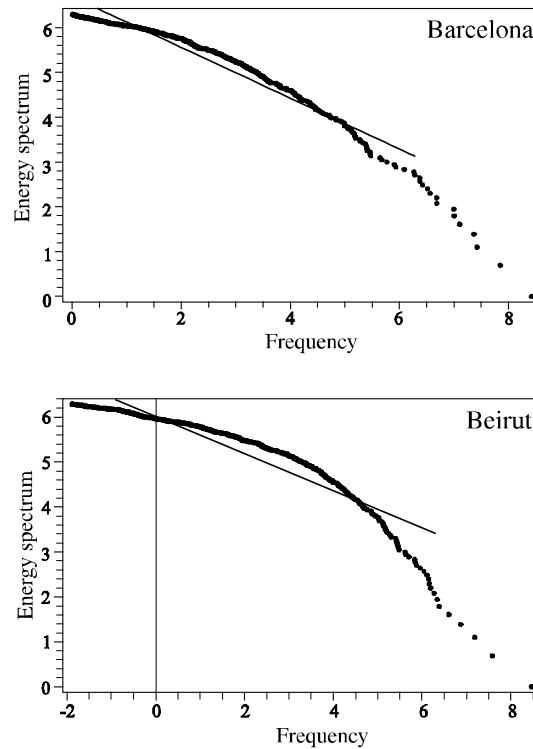
The Hurst coefficient, which is equal to 0.37, indicates an anti-persistence effect and therefore a high level of irregularity in these decadal rainfalls. In terms of rainfall, the decades follow each other but do not resemble each other. These are no successive dry and then humid periods, as in the Nile Valley. Note, however, that the pattern observed in Egypt was based on annual or monthly rainfall patterns, not decadal. Moreover, for monthly rainfall, the Hurst coefficient, which is equal to 0.55, indicates a persistence effect. This behavioral change between persistent and anti-persistent, based on resolution, should not be a surprise. This persistence is also now established in hydrology.

In order to appreciate the adjustment quality, the program provided in the Appendices also gives the determination coefficient and values for the Akaike and Bayes criteria. All of these indicators translate into a mediocre adjustment for the three tests, whether they are for raw decadal, differentiated or monthly rainfalls.

The same approach applied with wavelet decomposition gives a similar result, with a slope of -0.34 and a Hurst coefficient equal to 0.32. However, the test results are clearly better. The two energy spectrums are regular (see Figure 4.11). Furthermore, the Akaike and Bayes criteria are satisfied and the determination coefficient is raised to 0.90. These results are quite clearly of more interest when compared with other pluviometric series.

Similar results are obtained for Beirut with Hurst coefficients of 0.35 (Fourier decomposition) and 0.33 (wavelet decomposition). These values, which are very close to those calculated for Barcelona, reveal a profound resemblance in terms of irregularity, even if the depths of rainfall recorded are very different. This is not so surprising, since they are caused by Mediterranean disturbances of equivalent durations.

Although we have reached this stage of reflection, we are as yet unaware whether or not Barcelona rainfall is multifractal. In order to respond to this question, we must examine some ways of calculating fractal dimensions.



**Figure 4.11.** *Energy spectra from wavelet decomposition of decadal rainfall recorded for Barcelona and Beirut*

### 4.3. Some algorithms for the calculation of the fractal dimensions of auto-similar objects

Having classified their observations within a fractal category, in order to calculate the fractal dimension, geographers have to choose a suitable algorithm that conforms to these previous results. The following sections describe some of these, and highlight their applications and limitations.

#### 4.3.1. *Box counting, information and area measurement dimensions for auto-similar objects*

We begin with the most classical dimensions, relating to auto-similar fractals. When the phenomenon is an auto-similar fractal, based on the problem being analyzed, geographers choose to calculate a box counting, information or area

measurement fractal dimension. There are even more original algorithms that are used, for example, to determine the fractal dimension of a network, a digital land model or a town perimeter. Further interesting works on this subject include those written by J. Russ [RUS 94] and L. Seuront [SEU 10].

For each dimension, several algorithms are possible. This technical refinement, which is not always useful, is explained by the precedence of these fractals, which have been the subject of many studies. Only some *Mathematica* programs are included in the Appendices, since the majority of software applications available are relevant in determining the fractal dimensions of auto-similar objects. Most even extend beyond the importance that we attach to them in this book, which is of modest importance, since geographical phenomena are either self-affine fractals or multifractals.

#### 4.3.1.1. *Box-counting techniques*

The box-counting, or paving-stone, method is the most classical technique used. A set, such as an urban settlement represented by the value 1, which is separated from the non-urban background described by the value 0, is covered by a mesh of boxes of increasing length,  $r$ . The box-counting fractal dimension is then obtained using the formula:

$$Db = -\lim_{r \rightarrow 0} \log(N(r)) / \log(r) \quad [4.5]$$

where  $N(r)$  is the number of boxes necessary to cover the urban settlement.

This technique, which is often implemented in geography, has a number of faults that have been identified in numerous studies conducted by statisticians. For example, a preliminary threshold, which is sometimes too simplistic, needs to be imposed in order to separate the object from the background. The information handled is reduced to a binary coupling. Furthermore, the fractal dimension is dependent on the increasing size,  $r$ , of the boxes and even on the positioning of successive gridlines placed over the image being investigated. In addition, contrary to the underlying auto-similarity principle, the box-counting fractal dimension is dependent on resolution. Also, according to N. Kojima *et al.* [KOJ 06] the differences reach 23% for non-forested areas of Bolivia when the box-counting fractal dimension is calculated on images with a resolution of either 30 m, or 1,100 m. Finally, this fractal dimension is always a measure of the irregularity of an object, but its auto-similarity must be pre-verified.

Consequently, the first algorithm has had to undergo multiple improvements. For example, the box-counting method is applied to grayscale images.  $N(r)$  thus corresponds to the difference between the maximum and minimum grayscale values calculated for each box. It is also possible to choose the shape of the boxes, and even

their localization. B. Mandelbrot's *compass* or *yardstick* techniques also belong to this category of box-counting method, as well as the so-called "extended counting" technique in which the fractal dimension is calculated over different parts of the fractal object and the maximum value obtained is considered the "real" fractal dimension. It is not a surprise, then, that this approach underestimates the theoretical fractal dimension.

The box-counting method therefore encompasses a number of algorithms, as shown by the works of researchers from the Besançon and Strasbourg schools that form the basis of the *Fractalyse* software package. Furthermore, this software also calculates lacunarity and is useful for producing auto-similar fractal scans, but its effectiveness is, above all, relevant to auto-similar fractals, since this technique is only truly valid for this type of set.

#### 4.3.1.2. Information fractal dimension

The information dimension,  $D_i$ , is no less easy to obtain. The information or Shannon entropy is defined by the formula:

$$E_i = -\sum f_i \log f_i \quad [4.6]$$

where  $f_i$  is the frequency.

This entropy  $E_i$  is calculated over the entire series, then over the series obtained by dividing the initial series by 2, then 4, then 8, etc. By proceeding to the adjustment of the  $E_i$  values obtained, the information fractal dimension,  $D_i$ , is equal to the slope of the regression line. This type of algorithm is regularly used in medical or meteorological image processing.

Compared with the box-counting dimension, the information dimension offers two advantages. On one hand, it takes account of the mass of the objects concerned. On the other, the information fractal dimension,  $D_i$ , is equal to the box-counting dimension,  $D_b$ , when the objects are uniformly distributed over space. For a non-uniform distribution, however, it is less than  $D_b$ , whether referring to towns in a region, businesses in a town or sinkholes on a karst. It is therefore possible to measure the uniformity of the localization of a geographical phenomenon by comparing these two fractal dimensions and deducing an indicator of fractal agglomeration from them:

$$I_a = D_b - D_i \quad [4.7]$$

This approach, which we have never encountered in geography, has sometimes been applied in medical imaging and marine biology in order to compare the distribution of species in turbulent waters.

#### 4.3.1.3. *Area measurement techniques*

The so-called “area measurement” methods bring together a third set of techniques. They utilize structuring elements, such as triangles or mathematical erosion morphology, and dilatation operations on the perimeters and surfaces.

Among these numerous techniques, isorhythmic techniques were favored in the first geographical studies of fractals [LAM 93]. A fractal dimension is estimated for each isorhythm of a surface, for example for each level curve of a topographical surface, or each isohyet of a tropical region, or even the isoprice of a catchment area. Mostly, this fractal dimension is determined using the box-counting technique. It therefore presents the defects from it.

The surface fractal dimension is then obtained by averaging the estimated dimensions over all of the level curves, to which the value +1 is added in order to take account of the topological dimension of a surface. These methods are dependent on a number of level curves being used.

Some “white coverage” or triangle techniques were also being proposed in studies during the 1980s, but these tools are only valid for auto-similar fractals, whose exceptional characteristics are acknowledged.

#### 4.3.2. *A geographically inconclusive application from perception*

The calculation of fractal dimensions by applying the box-counting method to world maps from the point of view of the inhabitants of three capital cities does not give a very conclusive result. In effect, Muscovites perceive a world that is barely more irregular ( $Db = 2.57$ ) than the residents of New-York ( $Db = 2.56$ ) or Parisians ( $Db = 2.55$ ). However, the difference is clearer when testing a fourth map, taken from the point of view of the inhabitants of Sydney. Their world space, with its vast marine expanses, is more homogeneous ( $Db = 2.50$ ).

In spite of everything, the differences are not very significant and the calculation of other fractal dimensions would also turn out to be misleading. While the irregularity of these three perceived worlds is comparable, their form is quite different. It is not enough to describe all facets of the irregularity of a form using a fractal dimension. As these worlds in fact differ in terms of their textures, the calculation of a lacunarity indicator is more efficient.

Table 4.2 summarizes measurements of lacunarity calculated on perceptual maps of the world from the point of view of the residents of Moscow, New York and Paris. As this indicator varies based on resolution, the calculations were performed by varying the size of the convolution windows, from 5 to 100 pixels squared. As

the space taken up by oceans is extremely vast compared to that of the continents, lacunarity was very high for small textures, whichever capital city was considered. Then, the values of this indicator reduced when the texture size was increased, and therefore the size of the convolution window.

| Window side length (in pixels) | Moscow | New York | Paris |
|--------------------------------|--------|----------|-------|
| 5                              | 3.35   | 3.50     | 3.36  |
| 10                             | 3.15   | 3.27     | 3.15  |
| 30                             | 2.68   | 2.74     | 2.662 |
| 50                             | 2.40   | 2.43     | 2.28  |
| 100                            | 1.91   | 1.83     | 1.81  |

**Table 4.2.** *World perceptual map lacunarity for residents of Moscow, New York and Paris*

It is especially interesting to observe the inversion of the lacunarities. For New York it is at its maximum at large scales, and therefore the world perceived by its inhabitants is heterogeneous at large scales. Conversely, it is more homogeneous, still at these scales, for the residents of Moscow. At small scales, we get the inverse effect; with the residents of Moscow perceiving a more heterogeneous world than that perceived by inhabitants of New York. Muscovites and New Yorkers perceive two different worlds.

Furthermore, this example demonstrates that translational invariance, which is measured by lacunarity, is dependent on scale. In concrete terms, the heterogeneity of a texture is modified with scale. Lacunarity may therefore be an effective parameter for texture differentiation, but is it dependent on scale?

Finally, this example shows that it is always possible to calculate a fractal dimension, which measures the irregularity of a geographical phenomenon. The fractal, and therefore multiscale characteristics, of these perceptual maps, which are in truth very crude are still to be demonstrated.

A test is possible using lacunarity measurements. Indeed, when reporting on a log-log graph for an auto-similar fractal the points on the graph whose coordinates reflect the lacunarity values, and the window size used in its calculation follow a straight line with a slope equal to the topological distance less the fractal dimension. For the three capital cities, the log-log adjustment is more or less satisfactory, but none of the values of the three slopes strengthen the idea that perceptual maps are auto-similar fractals, an idea that is already perceived with a simple visual test.

#### 4.4. The fractal dimensions of objects and self-affine processes

##### 4.4.1. A multitude of algorithms

For self-affine forms, the choice depends on the type of underlying process. After the tests to differentiate the self-affine fractal type observed, it is easy to calculate the spectral fractal dimension, having already determined the Hurst coefficient,  $H$ . In some disciplines, however, notably in ecology, other algorithms are frequently used.

##### 4.4.1.1. Spectral fractal dimensions

The spectral fractal dimension,  $D_s$ , is given by the two formulae:

$$D_s = D_t + 1 - H \quad [4.8]$$

for the trace over time of fractional Brownian motion, and:

$$D_s = 1 / H \quad [4.9]$$

for the trajectory in space of fractional Brownian motion. For a simple Brownian motion,  $H$  is equal to 0.5 and it is straightforward to find the two fractal dimensions along the trace, which are equal to 1.5, and dimensions of the trajectory in space, which are equal to 2.

For the other Brownian models, it is sufficient to calculate  $H$  from the formulae summarized in Table 4.1. Thus, for an integrated fractional Brownian motion, the fractal dimension along the trace is:

$$D_s = D_t + 1 - ((\alpha - 3) / 2) \quad [4.10]$$

whereas for a differentiated fractional Gaussian noise, the following equation holds true:

$$D_s = D_t + 1 - ((\alpha + 3) / 2) \quad [4.11]$$

where  $D_t$  is the topological dimension, which is 1 for a transect or chronicle series, 2 for an image or map and 3 for a volume, such as a relief.

For a Lévy flight, it is necessary first of all to calculate  $H$  by applying equation [4.1].

Then other algorithms – which are examined in detail in the work of J. Lévy Véhel [LEV 02], which groups them into two families – are deployed in order to

determine the  $\alpha$ -stable distribution parameters, notably the exponents of stability and scale, from which the fractal dimension can be deduced. In a more concise article, V. Seshadri and B.J. West [SES 82] deduced the fractal dimension along the trace of a Lévy flight from the scaling parameter,  $\mu$  from the formula:

$$Ds = 2 - 1 / \mu \quad [4.12]$$

The attentive reader will appreciate the significance of the Hurst coefficient, which varies according to the process considered, in equations [4.9] and [4.10]. The same formulae are used from the energy spectrum issued from a wavelet decomposition. As the wavelet decomposition better fits the signal shape than the Fourier decomposition, the results obtained using this technique are considered to be mathematically more well founded.

#### 4.4.1.2. *Other singular fractal dimensions*

For a fractional, and therefore non-stationary, Brownian motion, other techniques are currently used – in particular window variance analysis. A series is decomposed into multiple subseries of the same size and the variance is then calculated for each of these subseries. When these estimated variances on windows that are becoming smaller and smaller follow a straight line, an adjustment on this line provides the slope from which the fractal dimension can be deduced.

It is also possible to calculate the fractal dimension from an auto-correlation function. In this case, geographers follow an approach similar to that used in wavelet decomposition. The type of fractal is deduced from the slope and then the value of the Hurst coefficient, from which the the fractal dimension is derived.

Conversely, other techniques are only applied to a stationary process [ZAN 05]. For example, for a fractional Gaussian noise, a variogram fractal dimension,  $Dv$ , is calculated from the formula:

$$Dv = (4 - \alpha) / 2 \quad [4.13]$$

where  $\alpha$  is the absolute value of the slope of the linear variogram adjustment. This variogram is estimated from the formula:

$$\gamma(h) = 1 / 2 E \{ |Z(x+h) - Z(h)|^2 \} \quad [4.14]$$

where  $Z$  is the variable, and  $h$  is the lag in time or space.

A related technique [BEZ 10] consists of first calculating the madogram from the formula:

$$\gamma(h) = 1 / 2 E \{ |Z(x+h) - Z(h)| \} \quad [4.15]$$

then, from this, determining the slope. The madogram fractal dimension is then given by the formula:

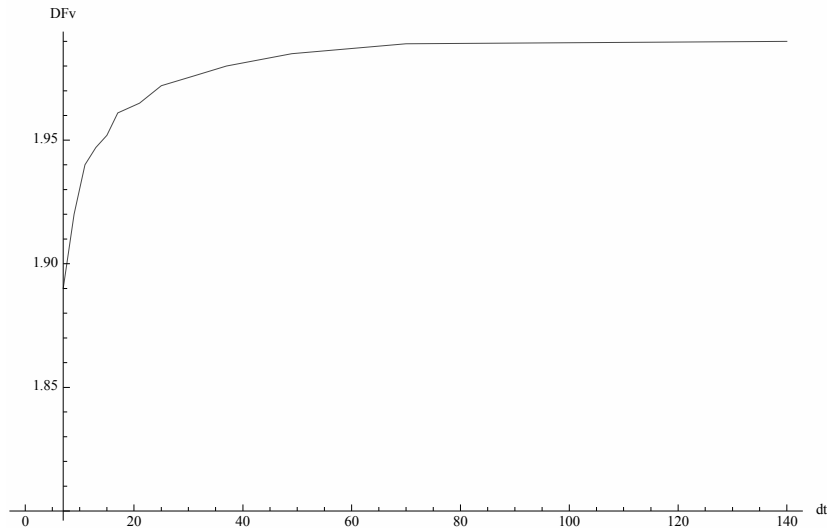
$$Dm = Dt + 1 - \alpha \quad [4.16]$$

This adjustment is made, in the variogram and madogram cases, on the first values relative to the origin, as long as they are increasing, without reaching the end of the range. This range depends on the geographical phenomenon studied, and it is understood that the calculation of the variogram or madogram fractal dimension are not recommended when the range is greatly reduced in time or space. This range should therefore be checked before proceeding with the fractal dimension calculation.

When the value of this range is below 3 or 4 in time for a chronicle series, the adjustment performed over such a small number of values is insignificant. In effect, these very short ranges in time or space indicate that the series being analyzed is non-stationary. Yet, the variogram fractal dimension uniquely does not undergo any bias for a stationary series [KUL 98].

More generally, the slope value  $\alpha$  depends on the number of intervals used, and therefore the variogram fractal dimension varies with the range [PAL 88]. It is possible to calculate this slope, and then the corresponding fractal dimension by choosing an ever-increasing number of intervals. The graphical representation of these successive fractal dimensions is described as a fractographic in ecology. This fractographic model therefore enables the variation of the fractal dimension based on scale to be studied. It is then possible, as with the Hurst parameter, to identify conditions of persistence or anti-persistence in time or space, which is not surprising since, for a series,  $H$  is equal to  $2-D$ . Such a study was undertaken on oyster beds by V. Kostylev [KOS 01]. The oysters were grouped into three boxed spatial structures. This technique, which was very popular in the 1980s, is less popular nowadays, despite its effectiveness in dealing with self-affine stationary series.

By way of an initial academic exercise, Figure 4.12 is a partial fractogram determined for differentiated decadal Barcelona precipitations. Fractal dimensions have been calculated for 7 decades, 9 and so forth, up to 140. The fractogram shows a constant increase in the variogram fractal dimension. Decadal rainfall recorded for Barcelona does not, therefore, present any embedded rhythms, although it is true that the differentiation performed in advance suppresses any seasonal trend and rhythm. This is a necessary condition for making the series of raw data stationary and therefore subject to fractal treatment using the variogram.



**Figure 4.12.** *Fractogram of variogram fractal dimensions for differentiated decadal rainfall for Barcelona*

It is also very easy to apply this approach to satellite images in order to separately analyze variograms and the corresponding fractal dimensions in opposite directions. Table 4.3 summarizes the results obtained for six Mediterranean ports. The variogram fractal dimensions and Hurst coefficients were calculated for the east–west and north–south directions for each satellite photograph. Each direction was represented by an average transect, defined from all of the east–west and north–south transects.

|                  | East–West  | East–West | North–South | North–South |
|------------------|------------|-----------|-------------|-------------|
|                  | <i>Dfv</i> | <i>H</i>  | <i>Dfv</i>  | <i>H</i>    |
| <b>Athens</b>    | 1.57       | 0.43      | 1.51        | 0.48        |
| <b>Barcelona</b> | 1.52       | 0.48      | 1.56        | 0.44        |
| <b>Beirut</b>    | 1.54       | 0.46      | 1.50        | 0.50        |
| <b>Genoa</b>     | 1.52       | 0.48      | 1.49        | 0.51        |
| <b>Marseille</b> | 11.51      | 0.49      | 1.55        | 0.45        |
| <b>Naples</b>    | 1.65       | 0.35      | 1.52        | 0.48        |

**Table 4.3.** *North–south and east–west variogram fractal dimensions and Hurst coefficients of Mediterranean ports*

These results give rise to a number of comments. First, with the exception of the Genoa north–south transect, all of the fractal dimensions are greater than or equal to

1.5 and the Hurst coefficients are less than 0.5. Therefore, the irregularities are generally greater than in a normal Brownian process. They are also stronger than for natural phenomena, for which we recall that the Hurst coefficient is close to 0.8. In the two agglomerations of Barcelona and Marseille, the heterogeneities are higher in the north–south than the east–west direction. The inverse is true for the other four ports, and these differences cannot be attributed to the linear coastal orientation. Finally, the heterogeneity of the urban tissue appears to be at its maximum in Naples, at least in the east–west direction.

It is also possible to use Hurst's R/S technique, which is harder to program, but enjoys a certain prestige due to its age. However the Hurst analysis is not very stable for short series. It requires at least 500 sets of data in order to generate non-biased results. More generally, the value of the Hurst exponent depends on the sample size. In addition, this technique only works with fractional Gaussian noise.

In conclusion, the best approach consists of calculating the fractal dimension from the slope of the spectrum of a Fourier decomposition or, better still, from a wavelet decomposition, having first identified the self-affine fractal type – whether it be a fractional Gaussian noise, a fractional Brownian process or a Lévy flight or walk. The fractal dimension, which is calculated from the variogram slope or from auto-correlation, is of interest as it is complementary to geostatistical approaches. Other techniques appear to be less well suited to geographical problems.

#### **4.4.2. High irregularity of decadal rainfall for Barcelona and Beirut**

The decadal and monthly rainfall series for Barcelona and Beirut, and no doubt for the majority of Mediterranean resorts, are fractional Gaussian noises. The spectral fractal dimensions deduced from equation [4.9] are equal to 1.63 (Fourier decomposition) and 1.67 (wavelet decomposition) for the rainfall series for Barcelona. For Beirut, these values are 1.64 and 1.76 respectively.

Comparing these values to those listed by P.A. Burrough [BUR 86], we note that these results are indicative of a high-level of fractality, which is normal since the fractal dimension is calculated on decadal rainfall, which is more irregular than the annual precipitations covered in the initial studies. The same calculations performed on the monthly Barcelona rainfall series gives a Hurst coefficient equal to 0.55 and a spectral fractal dimension of only 1.44. However, for these monthly rainfall series the Akaike and Bayes criteria remain above 325 and the regression model is therefore not very significant.

The spectral fractal dimension is barely any higher for Beirut. However, in these two cases, the Hurst coefficient becomes greater than 0.60, which is synonymous

with a persistence effect. Already at the level of the monthly rainfall series, phases of dryness are followed by phases of humidity. We see the pattern of the Nile flooding before the creation of the Aswan dam. While the choice of observation resolution, whether weekly or monthly, has little effect on the fractal dimension value, it does radically change conclusions concerning the persistence effect.

By applying a wavelet decomposition to the same series, however, the results force conflicting conclusions. Fractal dimensions, which are equal to 1.67, 1.78 and 1.87 for the basic and differentiated decadal rainfall series and the monthly rainfall series, are relatively higher. Above all, however, no persistence effect that appears with a Fourier decomposition is found at the monthly level. Complementary studies should therefore be undertaken in order to verify Hurst's findings, which are found in many publications but from studies over long series.

#### **4.5. Conclusion**

This first brief overview of the algorithms placed at the disposal of researchers is intentionally succinct. The results obtained from the experimental data are of an academic nature. For a more elaborate study, it would be advisable to work over a longer series and to proceed to more comparisons.

Despite everything, they demonstrate that the choice of a processing technique imposes a methodological thought process and preliminary tests that have been all too rarely performed in the masses of publications on fractals in geography. The most common error is when an algorithm applicable to an auto-similar fractal is used for a geographical phenomenon that, as a general rule, is a self-affine fractal.



## Chapter 5

# The Fractal Dimensions of Rank-size Distributions

“How can there be evolution if everything is in balance?  
By definition, systems in balance go nowhere”

Per Bak, 1996

This chapter explains some of the algorithms for calculating the fractal dimensions of frequential distributions. Among these, the rank-size distributions are the most frequently discussed. Identifying these distributions is a first stage in most scientific research. They provide measurement for numerous phenomena, for example:

- the magnitude and energy from earthquakes in seismology;
- the intensity and height of rainfall; and
- especially the hierarchy of urban agglomerations in geography.

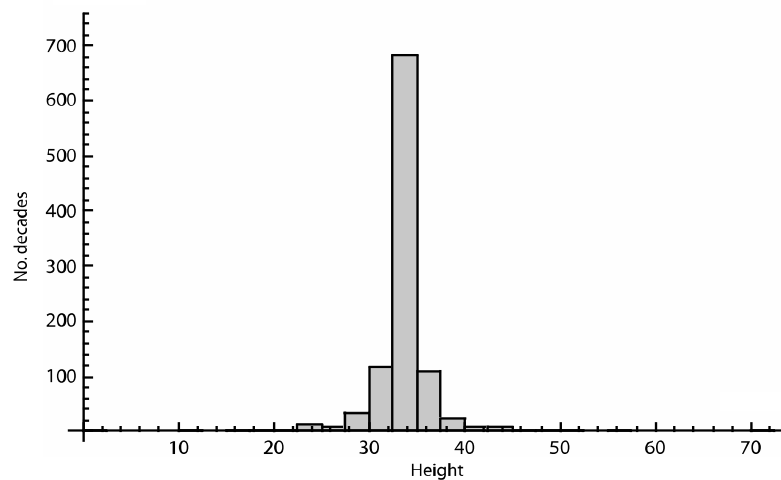
These distributions form the basis of numerous articles and research publications and they will be explained further in Chapters 7 and 8.

This chapter comprises three parts; section 5.1 gives a brief presentation of the data used in case studies. Section 5.2 examines the relationships that exist between

the Zipf, Pareto and power laws, which serve to determine the scaling parameter, which is the basis for the fractal dimension. As experimental data however rarely follow a perfectly straight line graph, the curves observed at one or both extremities, or even along the whole curve, force the selection of the best adjustment. This is the subject of section 5.3.

### 5.1. Three test series: rainfall heights, urban hierarchies and attendance figures for major French museums

In order to demonstrate the interest in the fractal approach as applied to frequency analyses, we are testing the techniques presented in this chapter on three data sets. First, in order to enable comparative studies, we have chosen the decadal and monthly rainfall for a number of Mediterranean resorts (Barcelona, Marignane and Beirut), which we have used in Chapter 4. In this chapter, however, these data are no longer considered according to their temporal rhythm, but are grouped into classes of recorded rainfall, as illustrated by the summary histogram in Figure 5.1, which outlines decadal precipitation.



**Figure 5.1.** *Histogram of decadal rainfall (in mm) for Barcelona*

A second set of data, of the same type, quantifies urban hierarchies in Italy and France with the help of the total population of the towns. We have, in fact, constructed three sets of data from this:

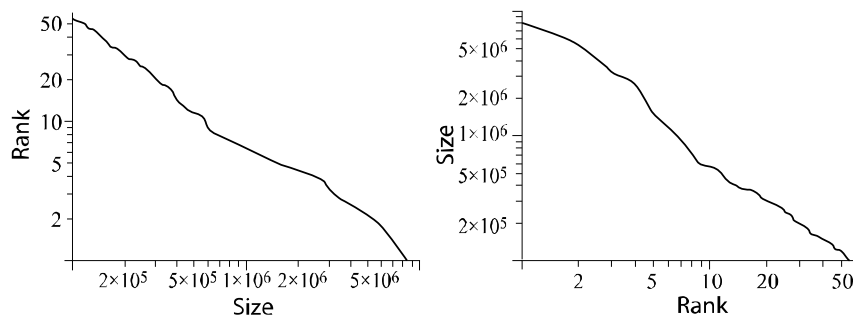
- the first is the set of all communal, rural and urban populations in France and in Italy;
- the second only keeps the 256 most populated communes, and therefore the most significant towns;
- for France, we deal with agglomerations of over 100,000 inhabitants.

Finally, in order to demonstrate the interest in the test explained in this chapter, a third data set quantifies attendance figures for the main French museums. This series, out of choice, only integrates those museums with over 100,000 visitors in 2007. The most visited, the Louvre, welcomed 8,222,000 art lovers; while the Augustin museum in Toulouse, with 100,072 visitors, had the least in this series.

## 5.2. The equivalence of the Zipf, Pareto and Power laws

It would be easy to provide endless examples of these laws, including those from physical geography, since for many geographical phenomena researchers make use of data in the form of frequencies. This is notably the case for French departments classified according to their tertiary population, mountainsides grouped into multiple classes according to their slopes, or visitors to religious or patrimonial sites classified according to the number of the faithful who frequent them.

By arranging these data in increasing or decreasing order, it is easy to construct a Pareto or Zipf graph (see Figure 5.2). Yet these two graphs are equivalent representations of a power law, for which the close link to fractals was emphasized in Chapter 1.



**Figure 5.2.** *Zipf and Pareto representations  
(attendance figures for the main French museums)*

Zipf's law, which is sometimes called the zeta distribution, presents frequency or rank on the y-axis and size on the x-axis, for example, a town's population. For Pareto's law, the two axes are inverted: the y-axis represents size and the x-axis, frequency or rank. These sets of data are represented on graphs for which the coordinates for both axes are expressed in logarithms, hence the name log-log or bi-logarithmic graph.

Thus, in order to determine the slope coefficients,  $\beta$ , the Zipf model is expressed as:

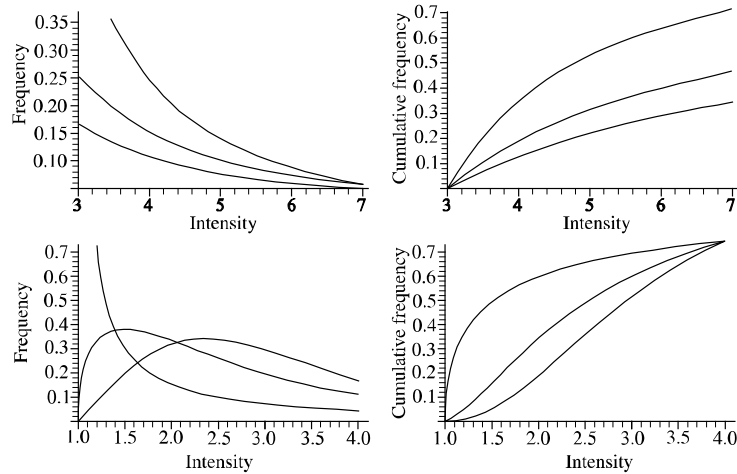
$$\text{LogRank} = \log A - \beta \log \text{Size} \quad [5.1]$$

while for a Pareto model that is sometimes called the Lokta model, the equation becomes:

$$\text{LogSize} = \log A - \alpha \log \text{Rank} \quad [5.2]$$

Consequently, the scaling coefficient of the Pareto law,  $\alpha$ , is equal to  $1/\beta$  while the scaling parameter of the power law is equal to  $1+\alpha$ . For a strict Zipf law, the scaling parameter is therefore equal to 2.

In addition, Zipf's law deals with discrete data, while Pareto's law is used in order to continually adjust variables partitioned into classes. Furthermore, Zipf's law is less general, or at least it was until its extension by Mandelbrot or Benini, since the slope,  $\beta$ , of the adjusted straight line graph should be equal to 1.



**Figure 5.3.** The Pareto I and Pareto IV distribution laws  
(source: produced using Mathematica software)

Zipf's law is therefore only a particular case of the Pareto laws. The plural is necessary here, since there are several groups of Pareto laws. The Pareto I distribution is characterized by a minimum value and a form parameter. A localization parameter is added to these to obtain a Pareto II distribution. Finally, another form parameter is included for a Pareto IV model. The Pareto III law is a borderline case of the Pareto IV law, which is quite rare in practice.

The visualization of the absolute frequency curves and cumulative frequencies (see Figure 5.3) of the Pareto I and Pareto IV models distinguishes these differences well. The bottom two graphs, which represent the Pareto IV distribution, reveal a more flexible model than Pareto I, which is outlined by the graphs at the top of the figure.

Finally, in Pareto formalism, frequency is represented in the form of cumulative frequencies, since the economist Pareto was himself interested in revenues above a certain threshold. This comment is not without significance, as statisticians prefer to perform their tests and adjustments on cumulative, rather than simple, frequencies. The shape of the simple frequency curve is highly dependent on the choice of class intervals. A poor choice of interval leads to visual errors and then errors in calculation. This badly thought-out choice of class intervals therefore has repercussions on the determination of the scaling coefficient, and then the fractal dimension.

This problem is well known and there are many technical solutions to be found in software dealing with statistical treatment or automated cartography. Thus, using *Mathematica*, geographers can construct the histogram by choosing techniques developed by Sturges, Scott, Freedman-Diaconis or Wand, based on the data that it handles. It is even possible, as proposed by C. Brown and L. Liebovitch [BRO 10], to build frequency curves using unequal class intervals selected by the user. Another solution that is frequently implemented consists of choosing logarithmic class intervals. Despite this veritable artist's palette of solutions, most statisticians believe that it is better to work on cumulative frequencies or probabilities.

The Zipf and Pareto laws, and more generally the power laws, have been known since well before the advent of fractals. They have been, and remain, the subject of countless articles. They are notably chosen in order to adjust the heavy tails of  $\alpha$ -stable distributions. These distributions are asymmetrical and their tails decrease according to a power law, and not an exponential distribution as with a Gaussian curve. Geographers encounter  $\alpha$ -stable distributions in the majority of studies on risk. Unlike the learnings from a normal distribution, heavy tail distributions signify that extreme events have a non-negligible frequency (see Chapter 9).

### 5.3. Three strategies for adjusting the rank-size distribution curve

Many technical precautions must be taken in order to avoid errors that are occasionally committed. Generally, these frequencies do not follow a perfectly straight line graph, but present a fairly pronounced curve, which is induced either by small and very frequent events or by events of exceptional magnitude, such as the over-dimensioned capital cities and the innumerable smaller cities in the case of the sequential distribution of towns according to their population. In other cases, we are referring to fairly significant noise.

The examination of rank-size distributions for towns effectively shows that in all countries there are more small towns than large ones. They produce a curve for which the amplitude depends on the population threshold used to define the town. When a town contains 10,000 or more people, as in Spain, or even 50,000 in Japan, the curve is less pronounced than in France, where the same threshold is set at 2,000 inhabitants. This curve becomes much clearer when all human concentrations from hamlets and villages are integrated into the analysis. G. Zipf had already noted this deformation for the distribution of towns in Australia.

In comparative studies, it is better to handle data that have been previously homogenized, but this has not always been the case. Another solution consists of handling delimited urban areas using night-time satellite imaging [SMA 10]. Global maps are available online and have the advantage of facilitating diachronic analyses, after some modest corrections to take account of the diversity of successive captors.

Regarding the primatial fact, which is always quoted in studies of French towns, this fact is without doubt very general around the world. F. Moriconi-Ebrard [MOR 98] even discovered a law about it, known as the law of metropolization: the ratio between the first and second town will be approximately 4.8. This law is also verified across all scales, for big and small States. It is therefore scale invariant.

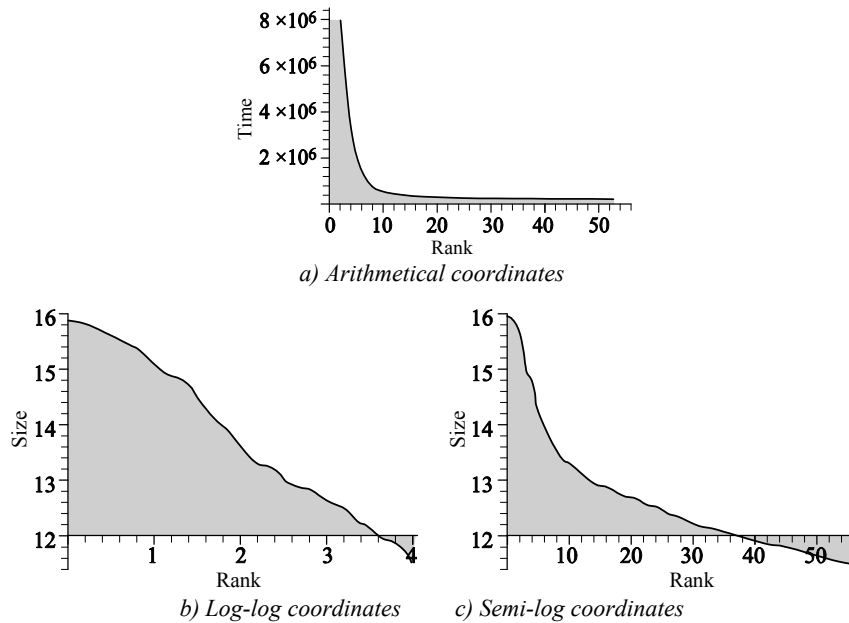
Thus the curve is applied, at least, at the two extremities, to the rank-size distribution of urban populations. More generally, taking this curvature into account has given rise to three strategies.

#### 5.3.1. *A visual approach using graphs*

The first strategy consists of analyzing all frequency series by drawing at least two graphs. The first graph is produced from arithmetical coordinates, but has a major drawback in that it does not enable a clear difference between an exponential or log-normal distribution or a power law to be established. On such a graph, these three distributions are poorly differentiated and so this is a mediocre test of the

fractality of a series. In exchange for this, deviations relative to the distribution can easily be perceived. It is therefore easy to spot secondary mechanisms, even if they are only random noises that affect the distribution order.

The second graph, represented using log-log coordinates, has an inverse visual effect. It masks anomalies, but discriminates better between the different distributions. On a log-log graph, a power law is represented by a straight-line graph; whereas an exponential distribution is defined by a curve. An exponential distribution only becomes a straight-line graph on a semi-logarithmic graph. The only disadvantage here is that this visual test is not very good for distinguishing a power law from a log-normal distribution. Yet, we are referring to two different random forms, the second of which is not synonymous with fractality.

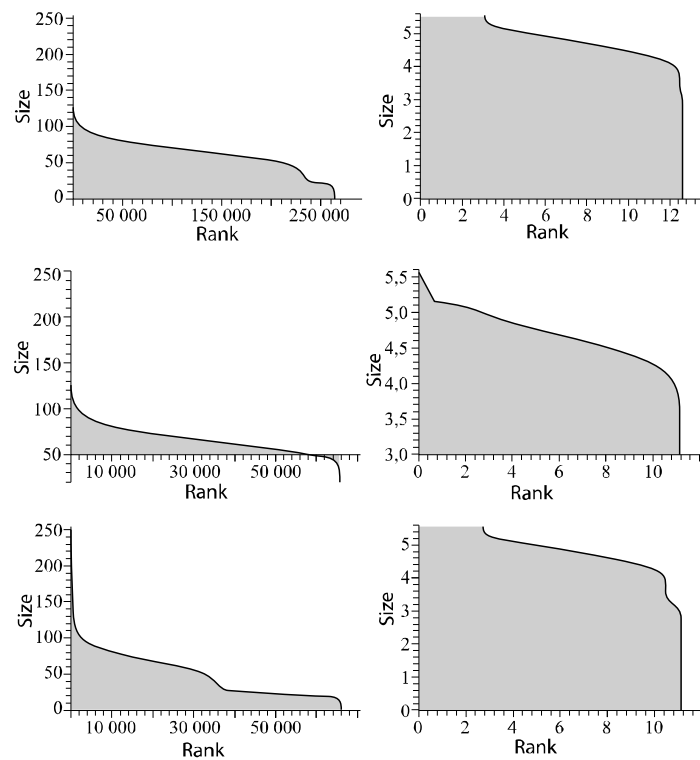


**Figure 5.4.** Visual tests on the series of museums classified according to their attendance figures

Figure 5.4 illustrates this visual test which has been performed for French museums that have been classified according to their attendance figures. We note that the hypothesis of a semi-log adjustment, and therefore of an exponential distribution, can be discounted, and that a power law is more appropriate. The frequential distribution of these great museums, which are classed according to the number of visitors, is highly fractal.

This visual strategy is explained in detail in L. Seuront's book [SEU 10], which examines examples of Zipf's laws that have been deformed by various secondary mechanisms. It explores cases of a power law that have been contaminated by an increasingly raised Gaussian noise, then by a semi-fractal, and therefore by the combination of the two power laws.

Finally, these visual approaches can be applied in two dimensions to images or maps. In two dimensions, three approaches are possible. The first consists of applying this approach to the transects of an image or to a digital land model, and then comparing the results. However, it is also possible to produce graphs for the whole image, which will be described by only two or three global graphs. Thirdly, there is nothing to prevent this approach being coupled with a scanning technique, and therefore to handle subimages and examine how these graphs are deformed based on the size of these subimages. Geographers thus make use of graphical multiscale analysis tools.



**Figure 5.5.** Visual Zipf model for the city and sectors of Barcelona

Figure 5.5 gives an example of this approach applied to the city of Barcelona. The two top graphs correspond to the whole city, while the graphs that follow model Zipf's law for the quarters of the original image. Only two quarters are represented, relating to the dense urban tissue and the coast and port quarter, for which, at the base of the figure, noise phenomena imposed by the land-sea disconnection show through. A more scientific extension of this uniquely visual exercise would class these subimages according to the significance of noise or deviations relative to Zipf's general law.

This visual approach was extrapolated by Y. Caron [CAR 04]. For each section of images, he established the Zipf curve using log-log coordinates, but also Zipf's inverse law curve, which was, in fact, a Pareto model, but constructed over frequencies and not cumulative frequencies. The joint utilization of these two tools enabled him to measure the quality of a compressed image, to detect an artificial object within a natural environment and to identify a zone of interest within an image.

### 5.3.2. *Adjusting the only linear part of the curve*

As in all visual analysis, these tests are necessary but insufficient as they are imbued with subjectivity. Pareto had adopted a second strategy. He applied his law, not to all revenues but just to the highest. He did not take low revenue, which was the most common and led to a curve on a graph that represented all the data, into account. This strategy eliminated a section of the data and only used a linear portion of the curve in order to proceed to a power law adjustment and then to determine the rank fractal dimension,  $D_r$ . This solution forces the threshold to be set.

There are numerous techniques available for defining the value of this threshold, but these are outside the context of this book. Moreover, the threshold set is frequently arbitrary [BEN 07]. Generally, we compare the quality of successive power models by modifying the threshold at each attempt using a Kolmogorov-Smirnov test [CLA 09], which comes down to indirectly drawing a dividing line between what is fractal and what is non-fractal.

This process was implemented in the last section of M. Forriez's thesis [FOR 10a], which was partly devoted to world cities. It proposed a cut-off at a threshold of 144,300 habitants that, in a geographical sense, is obscure and difficult to explain.

In mathematical terms, the law of scale is said to no longer be uniform, but to be asymptotic. It is no longer valid for all data, but just one section of it. This type of distribution is more common than uniform laws of scale. When this first choice is

made, the rank fractal dimension,  $D_r$ , is equal to the inverse of the slope obtained through adjustment, but in the majority of studies reasoning is based on the scaling parameter.

### ***5.3.3. Choosing the best adjustment, and therefore the most pertinent law***

A third strategy consists of verifying that the phenomenon being studied is a good match for a power law and not for any other distribution, notably the log-normal distribution since this is not synonymous with fractality. It is not scale invariant. This log-normal distribution, sometimes called Gibrat's law [GIB 31], was, however, regularly used in urban geography.

We saw above that a visual representation of raw values and values translated logarithmically is an indispensable, but insufficient in the first stage as the graphs of these laws are very similar. We have already recalled that the choice of class interval is a potential source of bias, which is all the more true when we settle for a visual approach. This is why this type of strategy, which is aimed at retaining the best adjustment and therefore the most relevant distribution, was recommended very early on, notably by M. Petruszewyck [PET 72], who emphasized the difficulty in choosing between Pareto's law and the log-normal distribution.

In order to design the best adjustment, the least squares method followed by a comparison of the determination coefficients is not sufficient. It is, however, used in the vast majority of studies, yet introduces many biases. For example, it leads to a highly exaggerated rejection of Zipf's law, and this is even more the case when the sample is small in terms of data. M. Barbut [BAR 88] demonstrated that a good adjustment of the logarithms could hide a very poor adjustment in non-transformed observations. He underlined that most geographical works on the rank-size distribution of cities were full of errors.

To avoid these errors, various corrections have been suggested. The most simple of these consists of subtracting a constant value, equal to 0.5, from each rank value. This approach is favored in biology and ecology, but it is advisable to proceed to an adjustment using the most reliable and most probable method. Also, the comparison used for retaining the best adjustment model relies on the two tests of Akaike and Bayes criteria, since the determination coefficient almost always provides good results, whatever the type of adjustment tested. We will observe them again in examples analyzed later in this chapter.

M. Dimou and A. Schaffar [DIM 07] compared these adjustment methods using a specific case. They demonstrated that the network of towns in the Balkans closely follows Zipf's law when the adjustment is produced using the maximum likelihood

estimates technique. This is not the case when the least squares method is used. These procedures are, therefore, complicated, except when using adapted software such as *Mathematica*, *Matlab* or their rivals *Open Source (Scilab)*. The programs, which are grouped together in the Appendices, therefore make an adjustment based on the maximum likelihood estimates technique, and then move on to these tests.

Applying this logic, there is nothing to prevent testing on the validity of multiple models over the series of data analyzed. Many propositions have been made in this regard [CLA 09]. Thus, in order to adjust the rank-size distribution of Spanish communities classified according to their populations, J.M. Sarabia and F. Prieto [SAR 09] envisaged four models: the Pareto, Tsallis and log-normal distribution models, as well as the so-called Pareto stable distribution, which is a generalization of the Pareto and Zipf laws and is given by the cumulative probability function:

$$F(x) = 1 - \exp(-\lambda(\log(x / \sigma))) \quad [5.3]$$

where:

$$F(x) = \Pr(X \leq x) \quad [5.4]$$

where  $\sigma$  is a size parameter equal to the population of the smallest community, and  $\lambda$  and  $\nu$  are two form parameters.

When these parameters are both equal to 1, we have a Zipf model and, more generally, we have a Pareto model if just parameter  $\nu$  is equal to 1. Graphically, this model is drawn as a straight-line graph when the two axes are expressed double logarithmically; whereas for a Pareto or Zipf adjustment, the two axes are simply logarithmic.

The stable Pareto model provides a better adjustment over all populations, both urban and rural in Spain, as shown by the maximum values for the Akaike criteria. Based on this, this model offers the advantage – which is also a disadvantage – of not producing a clear dividing line between the rural and urban worlds. Evidently, the calculation of the fractal dimension proceeds to the selected model.

Finally, another solution is forced when the curve is observed not only at the extremities but over all frequencies. It is therefore advisable to choose the parabolic fractal model proposed by J. Lah  rerre [LAH 98], and later spread throughout geography by P. Martin [MAR 04b] and his students [MAN 08]. The adjustment is produced using a parabola for which the function is written:

$$\ln(y) = \ln(\beta x) + \ln(\alpha x^2) + c \quad [5.5]$$

The parameter  $\beta$ , the curve coefficient, expresses the intensity of the curve: “It controls the inclination of the slope between the points” [MAR 04]. Parameter  $\alpha$  corresponds to the general slope of the distribution.

The choice of this model signifies that the fractal dimension is not stable. This adjustment has been compared with the linear adjustment of a power law, thanks to Akaike’s and Bayes’ tests.

Following the propositions of K. Rosen and M. Resnick [ROS 86], F. Guérin-Pace [GUE 93] had already proposed this type of adjustment for French towns, but without making any reference to the fractal approach, and therefore to the multiscale characteristics of urban hierarchy.

In conclusion, there are two main ideas to be considered:

- First, the adjustment law of the rank-size distributions is highly dependent on the data sample. If the rank-size distribution is occasionally log-normal for a set comprising all data, for example the population of all French communities or the number of employees for all companies, by only retaining towns whose populations exceed a threshold that varies by country, an adjustment using an applied Pareto law is better indicated. Thresholds of 500 inhabitants in Britain, but 10,000 inhabitants in the United States are sometimes accepted.

- Second, by only retaining very large phenomena, for example the top 20 or 30 metropolises in a state, a strict Zipf law or its applied equivalent, Bénini’s law, becomes possible.

Whatever the data treated, however, classical adjustments should be replaced by the least squares method using the sole determination coefficient as a decision criteria. An adjustment based on the maximum likelihood estimates method, complemented by the Bayes and Akaike tests, is recommended. These essential techniques must be followed, since the differences between log-normal distributions and Pareto-type power laws are barely perceptible. Yet, too many recent geographical studies are still proceeding to adjustments based on the least squares method.

#### ***5.3.4. Which rank-size distribution should be used for Italian towns, the main French agglomerations and all French communes?***

In order to define the best rank-size distribution for Italian towns, we used the *Mathematica* database. It is imperfect, since it is constructed from district data. For Italian towns, however, the very size of communes renders this constraint less significant. We also performed tests on French urban populations grouped into

agglomerations by the INSEE (Institut National de Statistiques et d'Etudes Economiques), which is more true to life. Then we tested the French and Italian communes together. Finally, this approach was applied to the decadal rainfall recorded for Barcelona and Beirut. More than the result, it was the method that mattered.

In order to test these series, we adopted the third strategy. No less than eight models were tested: log-normal, Zipf, Pareto I, Pareto IV, power law, Lévy distribution, extreme distribution law and parabolic fractal. Nothing was preventing us from testing other models, such as a stretched exponential distribution, which is the equivalent of a Weibull distribution function, or a shifted power law, as used by M. Gell-Mann [GEL 97]. The parameters for each adjustment were calculated according to the maximum likelihood estimates method, and then the program defined the determination coefficients, as well as the Akaike and Bayes criteria.

For 256 Italian towns, the determination coefficients of the eight adjustments were all above 0.93. Therefore, even if a geographer were to only work on a single model, the determination coefficient would signify to him or her that their choice was a good one. However, by examining the Akaike and Bayes criteria, it became possible to perform a selection of not just a good model but the best one.

Three models obtained the best scores. These were defined by their minimum values from the Akaike and Bayes criteria: the Pareto I, Pareto IV and parabolic fractal models. Choosing one from these was difficult, since all three displayed a determination coefficient above 0.98, and yet more similar values to the Akaike coefficient. There was only a slight gap between the three models, which may have been due to the communal nature of the data tested. The best solution consisted of making use of the simplest model, i.e. the Pareto I model.

| Distribution      | Determination coef. | Akaike criterion | Bayes criterion |
|-------------------|---------------------|------------------|-----------------|
| Pareto I          | 0.99                | -655             | -647            |
| Pareto IV         | 0.999               | -733             | -720            |
| Lévy              | 0.99                | -421             | -414            |
| Parabolic fractal | 0.99                | -179             | -169            |
| Power             | 0.98                | -137             | -128            |
| Zipf              | 0.98                | -136             | -128            |
| G.V.E.            | 0.97                | -136             | -129            |
| Log-normal        | 0.96                | -133             | -135            |

**Table 5.1.** *Comparative analysis of rank-size distribution adjustments using various probability distributions of French agglomerations*

We adopted the same approach, not on French town communes but on town agglomerations of more than 100,000 inhabitants. These are grouped in Table 5.1. The various adjustments were still very similar, at least when the determination coefficients were analyzed. They were all above 0.95. However, the gaps between the Akaike or Bayes criteria were very discriminatory. The two Pareto models were the best once again.

Finally, we repeated this exercise for all French communes, which were classified according to their population (see Table 5.2). Apart from the Lévy model, all of the other adjustments appeared to be consistent; the determination coefficients were again less discriminatory as they varied between 0.99 and 1. These extremely strong values were due to the large number of items in the data sample, all of which were positive. None of the other models appeared to contradict the choice of a log-normal model. Once again, however, the Akaike and Bayes criteria indicated that the Pareto model was the best choice.

| Distribution      | Determination coef. | Akaike criterion | Bayes Criterion |
|-------------------|---------------------|------------------|-----------------|
| Pareto I          | 1                   | -3076            | -3066           |
| Pareto IV         | 1                   | -3353            | -3336           |
| Lévy              | 0.97                | -155             | -145            |
| Parabolic fractal | 0.99                | -2090            | -2077           |
| Power             | 0.99                | -2047            | -2037           |
| Zipf              | 0.99                | -2047            | -2038           |
| G.V.E.            | 0.99                | -2048            | -2038           |
| Log-normal        | 0.99                | -2047            | -2037           |

**Table 5.2.** *Comparative analysis of rank-size distribution adjustments using various probability distributions of all French communes*

Before applying this strategy to Barcelona rainfall, we proceeded to a first-order differentiation, which had the result of removing any seasonal trends and cycles. With the exception of the Lévy model, all of the other adjustments were accepted, with determination coefficients above 0.98. An examination of the Akaike and Bayes criteria, however, gave priority to two models: the log-normal and Pareto IV adjustments. Unlike the previous examples, in this case it was very difficult to choose between two coincidental forms, no doubt because the series being studied only contained 1,024 pieces of data and because the resolution was decadal. The fractality of the precipitations was better dictated with daily or hourly resolutions.

These small exercises performed with two programs that are listed in the Appendices demonstrate two facts. First, on a methodology level it is imperative that

multiple models are compared so as not to settle for verification of suitability from a single model which nearly always turns out to adhere to the determination coefficient. Second, in terms of understanding, the hierarchical organization of large agglomerations follows a Pareto distribution or a parabolic fractal, rather than a log-normal distribution. This conclusion is without doubt valid for most countries.

If this little exercise is not convincing enough, then a read of an article written by C. Small *et al.* [SMA 10] demonstrates the interest of this strategy when using mathematical arguments. The initial data tested by these authors were on urban areas measured using night-time imaging by a meteorological defense satellite. This study effectively showed that adjustment distributions and parameters changed based on the urban areas selected, which are obtained by varying the luminosity thresholds in order to define an urban area. These researchers affirmed that the gradient of the rank-size distribution corresponds to an increase in connectivity of the network of agglomerations, which guides them on the trail of percolation models in order to explain the dynamics of urban hierarchies.

#### 5.4. Conclusion

This overview of the algorithms available to researchers studying frequential distributions is fairly well developed, since the rank-size distributions have remained the subject of intense studies with a view to better understanding the hierarchical organization of urban networks or all other geographical phenomena. The results obtained from these experimental data are, however, only of academic value. In more in-depth studies, it would be better to deal with more homogeneous series, to make greater comparative studies between states and to question the multifractal characteristics of rank-size distributions. We will return to these in the next chapter.

In spite of the limitations of these exercises, the third strategy comparing heavy tail models, that are therefore synonymous with autocorrelation and fractality, is by far the best. It enables us to declare that the Pareto I and IV and parabolic fractal models are best suited for arranging urban agglomerations according to their size. Zipf's model is only suitable for large agglomerations. The log-normal model is only acceptable when all communities are tested, as it makes no distinction between the smallest hamlets and global metropolises, which hardly conforms to a geographical approach. Furthermore, it is practically always less relevant than the Pareto models. We are emphasizing this point – which appears secondary – out of choice, since it challenges the traditional explanations of urban hierarchies that we will tackle in Chapter 7.



## Chapter 6

# Calculation and Interpretation of Multifractal Spectrums

“Insisting on the fact that reality is too complex to be encapsulated in a mathematical formula is nothing more than a form of obscurantism. Obscurantism and occasionally a defense against one’s own ignorance.”

Mario Bunge

This chapter explains a number of algorithms for calculating multifractal functions. It is comprised of four parts, beginning with a brief presentation of the data examined using this formalism. Section 6.2 sets out the approach to be followed in order to test the multifractal characteristic of a geographical phenomenon, whether it is a chronicle series representing a dynamic or a satellite image. Then, in section 6.3, we formulate some algorithms for calculating multifractal spectrums. We have engaged the simplest mathematical formalism in order to avoid bombarding readers with a long list of equations. Finally, section 6.4 is devoted to an introduction to the different approaches that have inspired the multifractal approach.

Before we start, however, we return to the essential question: what causes multifractal formalism? The fractal dimension is an overall indicator of the irregularity of a geographical phenomenon, river basin, urban area or series that is either frequential or ordered in time. To complement this, multifractals provide a

general but also local vision of all irregularities at all levels. Multifractals do not summarize information regarding “mean” irregularity, but they do provide all of this information. It is therefore possible to follow how coarseness, irregularities and disparities emerge, and are reinforced or disappear on various scales.

### **6.1. Three data sets for testing multifractality: a chronicle series, a rank-size distribution and satellite images**

In order to demonstrate the use of the multifractal approach, we apply the techniques explained in this chapter to three groups of data, most of which have already been called upon in previous chapters.

As the multifractal approach is applied to chronicle series, we have chosen two information series in order to make this first case work. We use two data sets already tested in previous chapters to find decadal and monthly rainfall levels recorded at several Mediterranean climatological stations (Barcelona, Marignane and Beirut).

In order to represent the frequential data, arranged not as a function of a time axis but based on their intensity, we have first studied populations of the set of Italian and French communes and second, studied the 256 most populated communes, and therefore towns in these two countries. These data, which are imperfect in terms of quality, come from the *Mathematica* “City” database. Conversely, the series classifying the main French agglomerations was too short to be of use as an example.

A third data set comprises Landsat 7 images of six port towns on the northern Mediterranean coastline: Athens, Barcelona, Beirut, Genoa, Marseille and Naples. This data set enables the production of high-quality comparative analysis.

As studies that have recourse to multifractal approaches are still uncommon in geography, notably in French geography, we are attaching more importance in this chapter to the handling of examples and the interpretation of results. Conversely, the presentation of algorithms only concentrates on the essential facts; it does not pretend to be complete.

### **6.2. Distinguishing multifractal and monofractal phenomena**

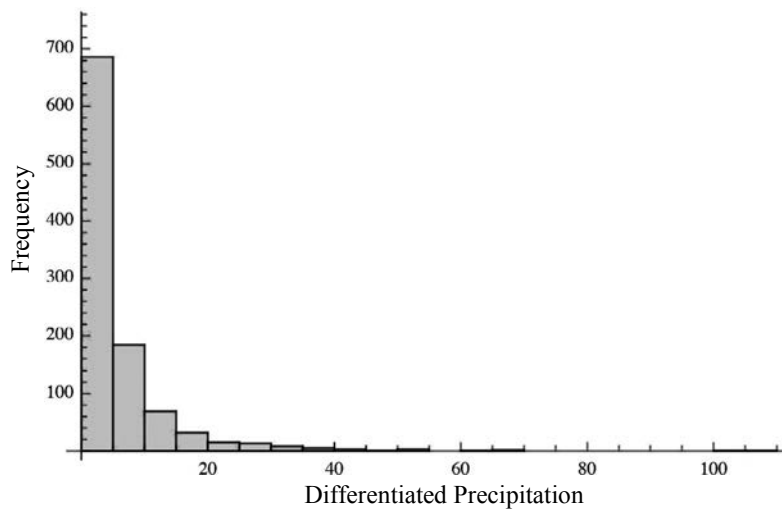
The first step in any multifractal study is to differentiate a multifractal from a monofractal. We recall that a *multifractal* is an application of fractal formalism. It is applied to a set represented by a measurement, for example a town that is represented by population densities or the number of business services by sector. In

order to differentiate a monofractal from a multifractal, geographers make use of two methods, visual and statistical.

### 6.2.1. *An initial imperfect visual test*

The first and simplest solution is a visual test proposed by O. Pont *et al.* [PON 09]. For a univariate series, such as a chronicle or frequential series, the histogram drawn up from absolute values from the differentiated series must display a slow decrease. Furthermore, the mode must correspond to low intensity in the differentiated phenomenon. These two characteristics are visually translated onto a histogram, revealing the presence of intermittence within the series being analyzed. Yet intermittence, which has been widely studied in research on turbulence, is an indication of multifractality. The same principle is applied to images and maps, but the histogram is outlined with modulus values of the image gradient.

Figure 6.1 illustrates the test applied to the differentiated decadal rainfall series for Barcelona. Undeniably, the mode of this series is smaller to the mean. There is also a good correlation with the lowest quantities of rainfall. However, the decrease is rather rapid.

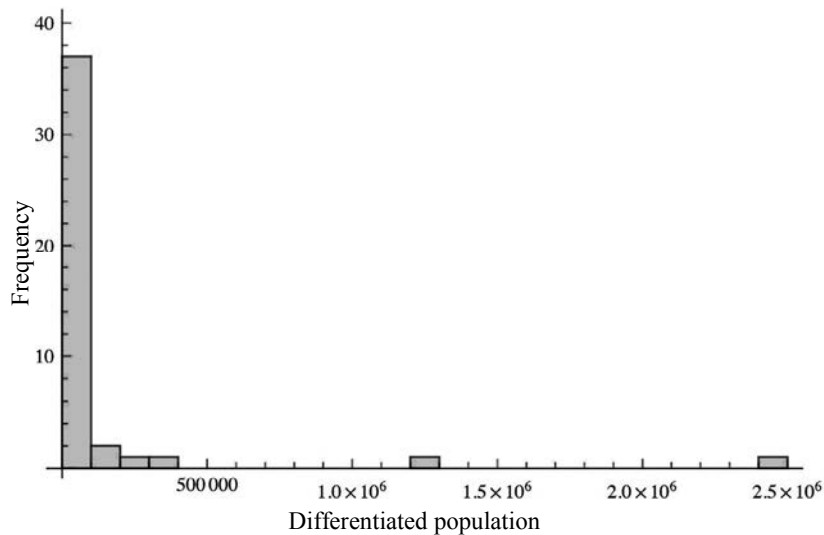


**Figure 6.1.** *Multifractality test on decadal rainfall in Barcelona*

Figure 6.2 translates the results of this practical visual test on the set of Italian towns, which are classified according to their populations. This graph is similar to

the preceding one and therefore merits the same commentary. The hierarchy of urban populations in Italy appears to be multifractal.

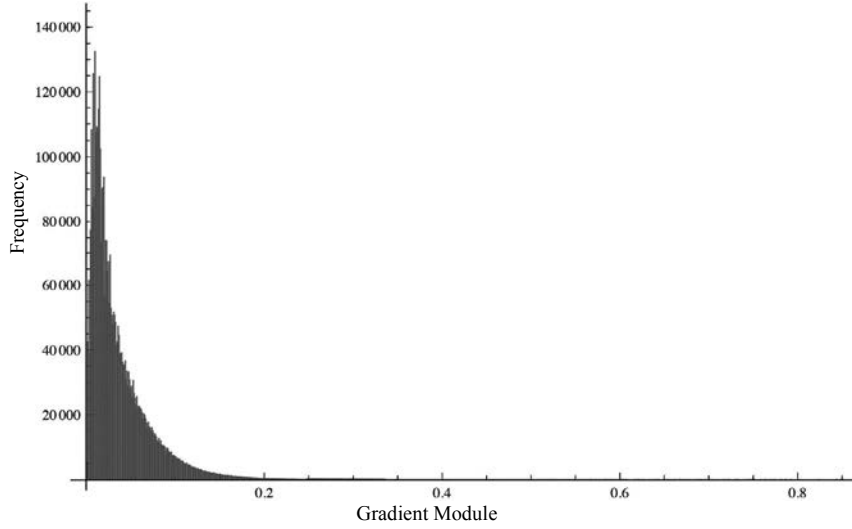
While a number of studies have demonstrated the multifractal characteristic of precipitations [BEN 97, BIA 04, GAR 08, HAL 01, HUB 93, LOV 95], however, far fewer attribute the same characteristic to the hierarchical organization of towns. To our knowledge only Y.G. Chen and Y.X. Zhou [CHE 04] have mentioned and analyzed this urban network multifractality. They moved away from theoretical reasoning, constructed over a Cobb-Douglas function, before applying the generalized Rényi correlation formalism that we tackle in section 6.2.1 of this chapter. This second test was applied to agglomerations in the United States and the results obtained were conclusive: the rank-size distribution of this country's urban hierarchy is multifractal.



**Figure 6.2.** *Visual multifractality test on Italian towns*

Figure 6.3 is the same visual test, but referring to two-dimensional space – the Landsat 7 satellite photograph of Barcelona. The data tested are the values in pixels, coded using 256 grayscale. The graph still displays a similar histogram to before, which is synonymous with multifractality. Moreover, the decrease is slower than in the other graphs, and so it is therefore possible to be more decisive and have greater confidence in attributing a multifractal trait to this Mediterranean port when it is described using a satellite image.

As with all visual tests, this test is imbued with a certain subjectivity. Moreover, we have hesitated to describe the shape of the decrease for the decadal Barcelona rainfall, which seems to change too rapidly. In order to confirm these initial results, some more objective tests need to be performed. These are explained below.



**Figure 6.3.** Visual multifractality test on the Landsat image of Barcelona

### 6.2.2. A second statistical test: generalized correlation dimensions

This second solution, which is statistical rather than visual, consists of calculating the generalized correlation dimensions,  $Dq$ , by using the formula:

$$Dq = \lim Iq(\delta) / \log(I / \delta) \quad [6.1]$$

where  $Iq(\delta)$  is the generalized information, also called Rényi information, which is given by the equation:

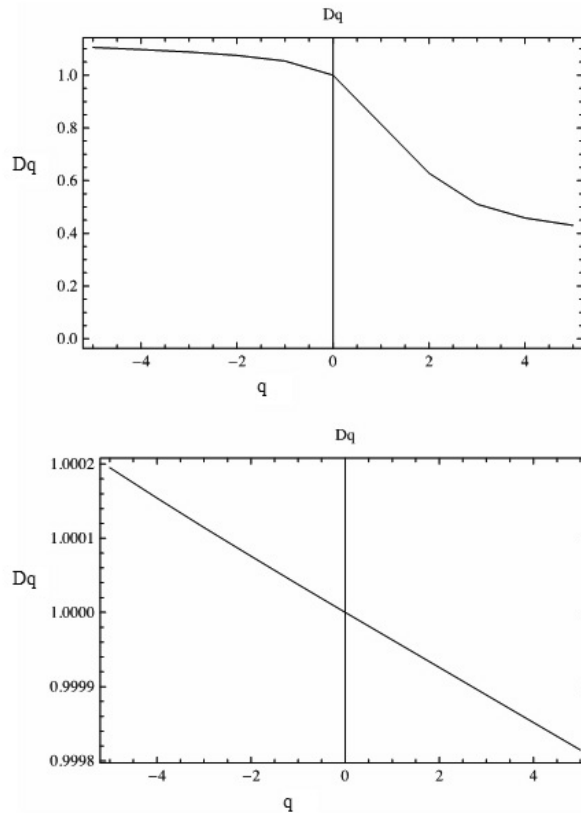
$$Iq(\delta) = (1 / (1 - q)) \log Mq(\delta) \quad [6.2]$$

where  $Mq(\delta)$  corresponds to the  $q$ -order moments of the frequency or probability distribution of the phenomenon being analyzed:

$$Mq(\delta) = \sum f_i(\delta)^q \quad [6.3]$$

Parameter  $q$  acts as a microscope. With positive values of  $q$ , the most irregular parts of the signal can be viewed, whether referring to a chronicle series or a set of measurements. Conversely, for negative values of  $q$ , the more regular or homogeneous parts of the curve or spatial area are observed. However, in order to avoid bias in the calculations, values for  $q$  are often between +10 and -10, and occasionally between +5 and -5. The graph of values for  $Dq$  is called the *generalized correlation dimension spectrum*.

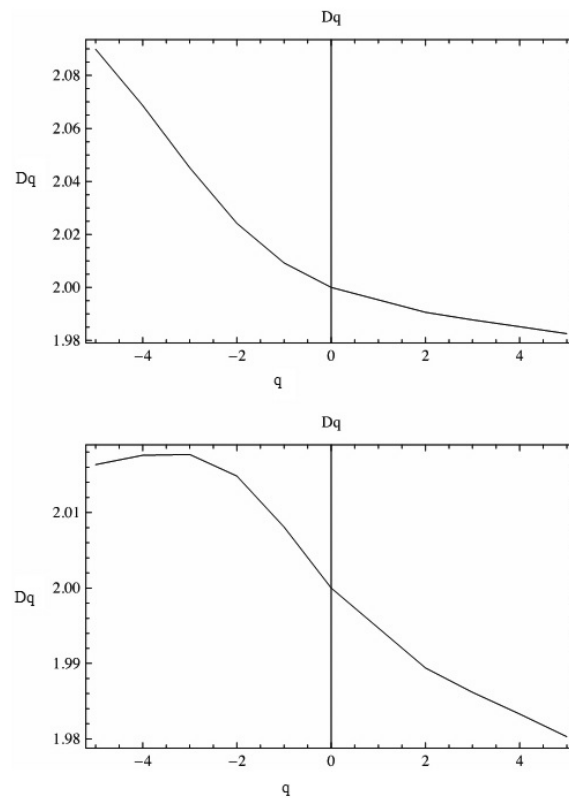
This spectrum (see Figure 3.6) illustrates the non-fractal (line parallel to the x-axis, corresponding to a whole number equal to 1 for a chronicle series or 2 for a map or image), monofractal (line parallel to the x-axis, but corresponding to a value that is not a whole number), or multifractal (decreasing logistical curve) characteristic of the series being studied.



**Figure 6.4.** Generalized correlation dimension spectrums for the 256 largest Italian towns (top) and the decadal rainfall for Barcelona (bottom)

Figure 6.4 represents the generalized dimension spectrum obtained for two simple series: populations of the 256 largest Italian towns and decadal rainfall levels for Barcelona. The graphs appear to indicate that the multifractal characteristic of the urban hierarchy in Italy is more assured than that of the Barcelona decadal precipitations, for which the spectrum is more linear than logistical. These results are consistent with the visual tests performed previously.

This hesitation concerning the decadal rainfall is without doubt linked to the choice of resolution: the decade. Another range, explored by A.P. Garcia-Marin *et al.* [GAR 08] consists of using hourly rainfalls. In this case, the temporal organization of precipitation is highly multifractal, at least in Southern Spain for the climatological station at Cordoba. Some similar results were previously discovered by P. Ladoy [LAD 93] for Nîmes. Thus, by using a more detailed resolution – hourly instead of decadal – the multifractal nature of rainfall can be better portrayed.



**Figure 6.5.** Generalized dimension spectrums for Athens (top) and Barcelona (bottom)

Figure 6.5 represents the same spectrums,  $Dq$ , but calculated on Landsat 7 satellite photographs of Barcelona and Athens. In both cases, the spectrums represent the form of a decreasing curve well, which is synonymous with a multifractal. Applying this algorithm to multiple northern Mediterranean coastal ports provides identical results.

This, a study of urban areas from a grayscale representation, and not from a simple binary image, is testament to the multifractal characteristic of large agglomerations. A single fractal dimension is insufficient to describe the irregularities of urban tissue represented in grayscale. Without doubt, this insufficiency explains the emergence and rapid spread of new tools, such as fractal scanning, which are highly valued by geographers confronted with the diversity of urban tissue.

Other spectrums are also used to verify the multifractal characteristic of an observed phenomenon. The so-called *structure spectrum* is often used. It is linked to the generalized correlation dimension spectrum using a simple formula:

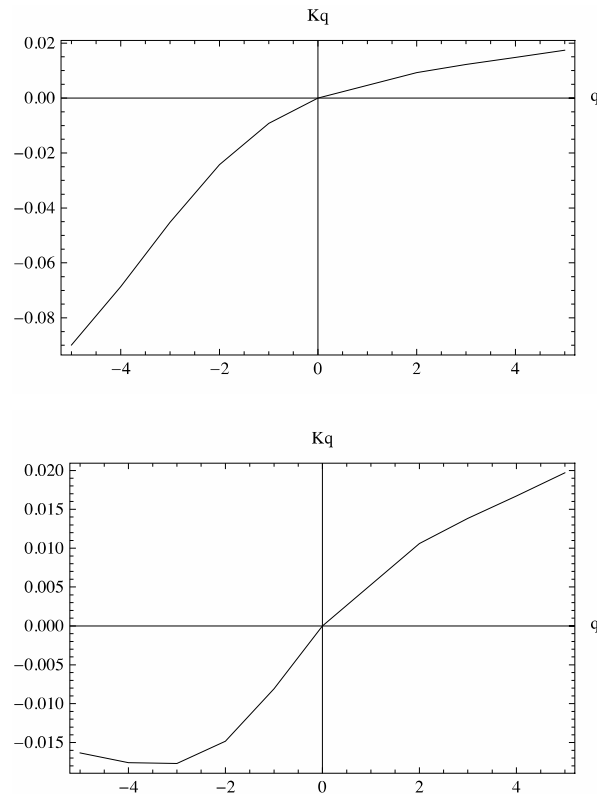
$$Kq = De - Dq \quad [6.4]$$

where  $De$  is the Euclidean dimension of the observation space.

On a structure spectrum, a monofractal is illustrated by a straight line graph, but a curve is indicative of a multifractal. Figure 6.6, which illustrates both structure spectrums for the port towns of Athens and Barcelona, confirms their multifractal characteristic when they are analyzed using grayscale satellite images.

All of the tests performed, whether visual or statistical, lead to the assertion of the multifractal characteristic of the examples studied. These results may appear surprising, but are easily explained. As soon as geographers analyze not just sets but sets described by measurements – whether referring to populations, social categories, rainfall intensities or grayscales – it is not unusual to observe multifractals, i.e. various objects where irregularities are propagated within all scales of the phenomenon being examined.

Indeed, in geography, as in all of the social sciences, an overlap of levels and interactions is the rule. Also, the results obtained following a multifractal treatment result in this universality of the coupling between multi-intensity and multiscale.



**Figure 6.6.** Structure spectrum for Athens (top) and Barcelona (bottom)

### 6.3. Various algorithms for calculation of the singularity spectrum

Having demonstrated the multifractal characteristic of a phenomenon, other spectrums are determined. The best known and most frequently used of these remains the *singularity spectrum*. It brings about not one single fractal dimension, but a function of fractal dimensions. As for the fractal dimension or the  $Dq$  spectrum, numerous algorithms enable singularity spectrums to be determined. Some of these are briefly presented in this book.

#### 6.3.1. Generalized box-counting and variogram methods

The first algorithms proposed were extensions of the *box-counting method*. They are still used in the majority of geoscience research articles available for

consultation. For two-dimensional studies, such as the analysis of maps or satellite images, geographers are able to use the software package *ImageJ* and its *FracLab* module. *FracLab* offers the benefit of great technical richness, the presentation for which is outside the scope of this work.

In addition, thanks to the works of A. Chhabra and R. Jensen [CHH 89] it is now possible to directly calculate this singularity spectrum without undergoing a Legendre transform, which commonly leads to errors. As with monofractals, however, these box-counting techniques promote bias. For example, these tools are only advised for isotropic fields, which are rather rare in geography. Finally, these techniques, which are acceptable in theoretical studies, are too inaccurate for applied studies requiring more precise values.

A second category of algorithms applies the *variogram method*. We explained this approach, which is highly valued by ecologists, in Chapter 4 in order to calculate the fractal dimension. By calculating generalized variograms, therefore replacing exponent 2 with exponent  $q$ , it is easy to construct a multifractal formalism inspired by the monofractal variogram model. This approach is detailed in the book by I. Rodriguez Iturbe and A. Rinaldo [ROD 97], although the same limitations apply as for the calculation of the simple fractal dimension. This approach is valid for single stationary processes, which are extremely rare in geography.

Finally, in order to calculate the singularity spectrum, other techniques have been derived from mathematical morphology, notably the “local morphological multifractal exponents” technique [XIA 06]. They are, however, only used in exceptional circumstances and we are not aware of any application of them in geography.

### **6.3.2. Methods derived from wavelet treatment**

Another category of algorithms follows on from wavelet decomposition. In Chapter 3 we saw the perfect adequacy of this technique and its application to multiscale phenomena in time or space.

The algorithms deduced from wavelet decomposition include a common first step: the calculation of wavelet coefficients. In this case, no meshing is required beforehand. These algorithms are therefore more efficient than techniques constructed from preliminary box counting. The second step – which was not included in early works based on wavelet decomposition – consists of only using parts of the wavelet coefficients; those encoding the strongest irregularities.

In the transform method, the *wavelet transform maximum modulus* (WTMM), which was the first method in widespread use in studies of turbulence or medical imaging, only used certain significant coefficients at each level of decomposition. This algorithm is available in the form of the *LastWave* program written by E. Bacry or modules within *MatLab*. This WTMM method is quite hard to implement, however, as it requires continuous wavelet decomposition, followed by skeletonization.

It is now preferable to use the *dominant coefficient method*, written in the *Matlab* programming language by P. Abry's team. This is well explained in H. Wendt's thesis [WEN 08]. First, it has the advantage of being more easily understood by non-mathematicians. This technique is also more precise for calculating the left part of the spectrum, which corresponds to negative values of the  $q$  exponents.

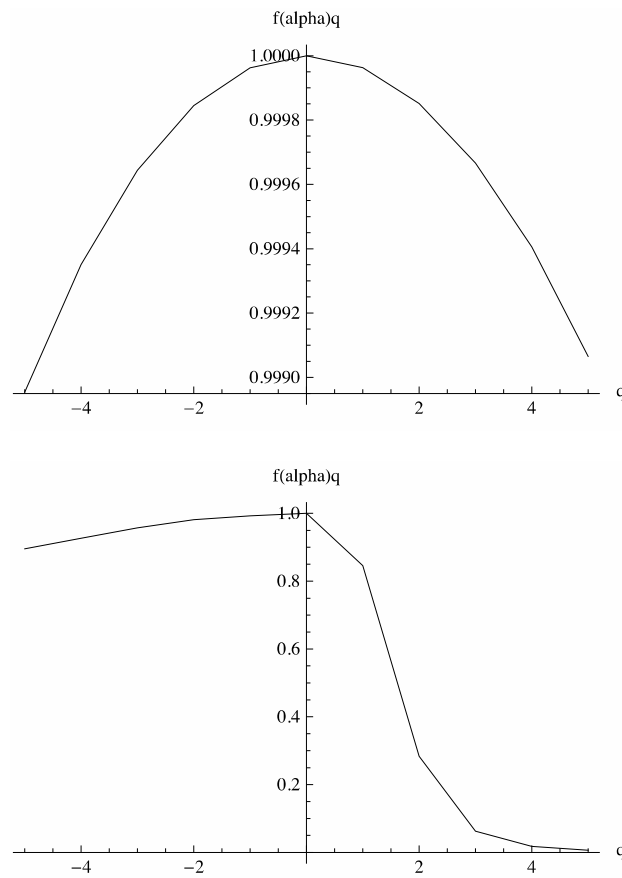
Furthermore, in order to appreciate the confidence limits of the spectrum obtained, *bootstrap* techniques are recommended by various authors [WEN 07]. Finally, calculation costs are reduced, as each technique begins with a discrete wavelet transform.

### 6.3.3. Interpretation of singularity spectrums

After these rapid forays into the technical world, it is advisable to interpret these spectrums. We first calculated them for decadal rainfall and large Italian towns, and therefore for two univariate series – a chronicle series and a frequential series – before applying this formalism to satellite images of Mediterranean ports.

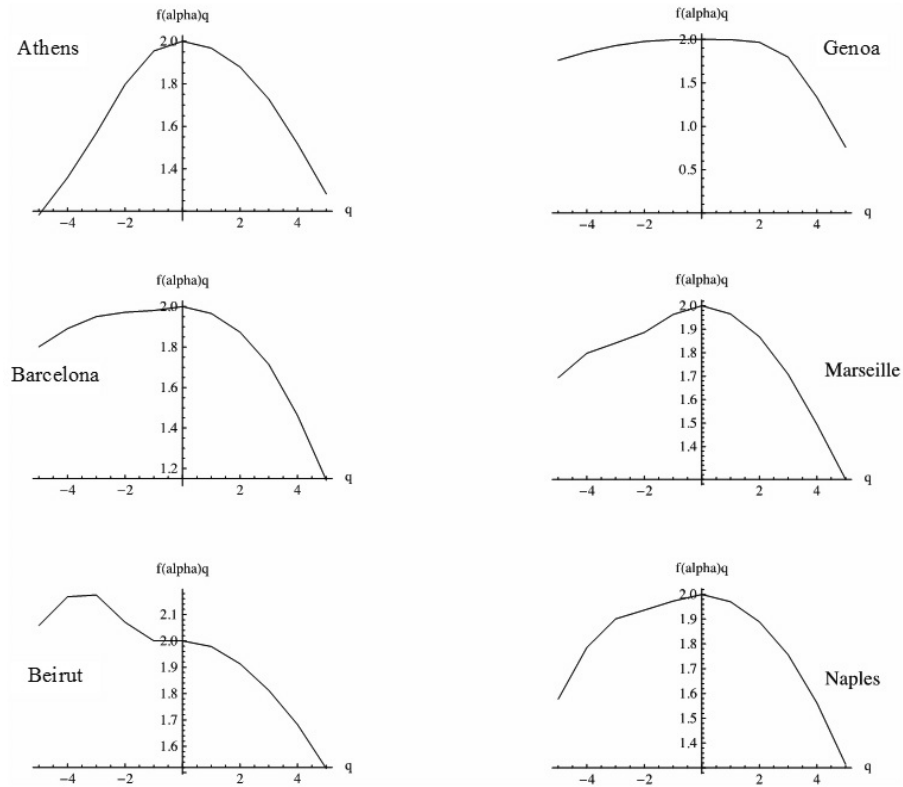
Figure 6.7 represents the two singularity spectrums for decadal rainfalls in Barcelona and the 256 largest towns in Italy. The decadal rainfall spectrum is perfectly symmetrical; irregularities, which are present at all levels, from 10 days to around 30 years, are regularly distributed in time and do not affect short sequences any more than long durations. Paradoxically, it is advisable to assert the regularity of irregularity in time and scale.

The opposite is true for the urban hierarchy of the 256 main towns in Italy; the spectrum is highly asymmetrical. It is possible that this asymmetry is partly the result of a bias in calculation. In spite of this, these irregularities, which are more significant on the left-hand side of the spectrum, indicate a more irregular hierarchy, notably within the small town classes. There is one piece of original information that should be verified with the other European urban networks. It is possible that the French town network should be distanced from this Italian model.



**Figure 6.7.** Singularity spectrum for decadal rainfall for Barcelona (top), and 256 major towns in Italy (bottom)

Figure 6.8 illustrates singularity spectrums calculated for six port towns: Athens, Barcelona, Beirut, Marseille, Genoa and Naples. All of these spectrums confirm the multifractal characteristic of these agglomerations, as they are not reduced to a single point.



**Figure 6.8.** Singularity spectrum for the ports of Athens, Barcelona, Beirut, Genoa, Marseille and Naples

The virtually symmetrical form of the spectrums leads to the agglomerations being split into three categories. Athens and Naples present irregularities that are spread across all scales. These are most often found on a large scale for the towns of Genoa, Barcelona and Marseille. For these towns, urban tissue irregularities are therefore higher when their reduced spaces are taken into consideration. The irregularities of these levels also become preponderant for Beirut, where the urban tissue is very finely irregular. The discontinuities within small urban structures are very strong for this agglomeration. On the other hand, irregularities are more regular at all levels for Athens, and are no more exceptional on a small or large scale. It should be possible to go into this analysis in greater depth by carrying out some automatic classifications.

Table 6.1 records the differences between the maximum and minimum values for each spectrum calculated for the six ports of Athens, Barcelona, Beirut, Marseille, Genoa and Naples and enables an initial overall comment. Genoa clearly stands out for its maximum multifractality, and thus highly rugged urban structure on all scales. In comparison, Beirut and Naples are the least fragmented agglomerations, even if strong irregularities characterize the urban tissue of Beirut, but only on a large scale. Athens, Barcelona and Marseille occupy intermediate positions.

| Port      | Fmax-Fmin | Port      | Fmax-Fmin |
|-----------|-----------|-----------|-----------|
| Athens    | 0.81      | Genoa     | 1.23      |
| Barcelona | 0.85      | Marseille | 0.73      |
| Beirut    | 0.65      | Naples    | 0.68      |

**Table 6.1.** *Difference between the maximum and minimum values of the singularity spectrum*

At first sight these results seem to be counter-intuitive, apart from Genoa, which is a multi-polar town with a center containing disparate small units and whose development is in contrast to its rugged terrain.

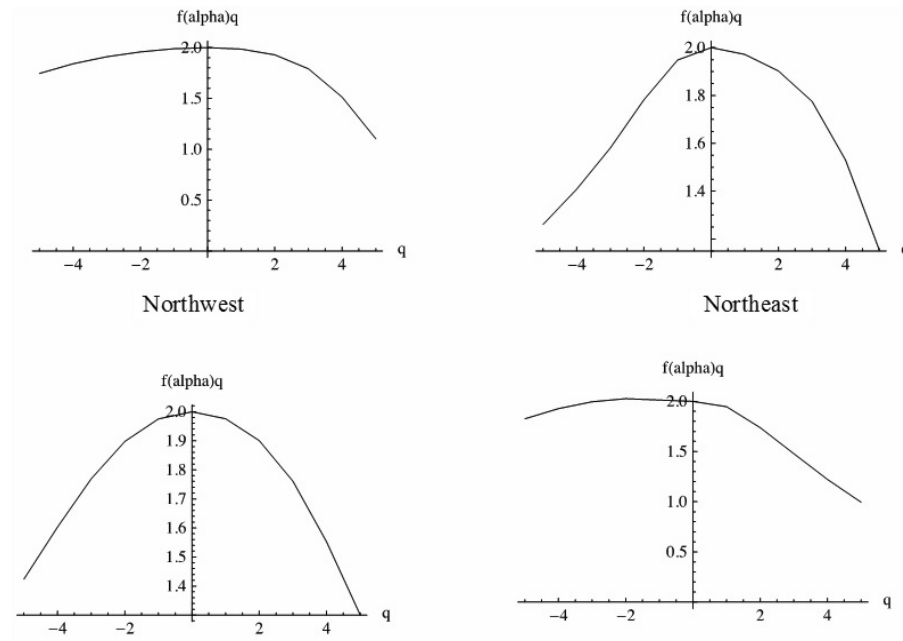
Let us not forget that these results are dependent on continuous information from satellite images and therefore indirectly on build-up. It is clear that a map showing the spread of social groups would give alternative results. However, this multifractal formalism would provide more reliable information on sociospatial segregation in these cities.

#### 6.4. Possible generalizations of the multifractal approach

This brief overview on the algorithms available to geographers is already complicated. In a number of connected disciplines, however, multiple generalizations have been put forward to improve the interpretation of these multifractal spectrums. Three examples merit some being mentioned, if not largely developed.

Firstly, several indicators have been deduced from various spectrums. For example, it is possible to calculate the difference between the maximum and minimum values of the various spectrums, which we have proved above for the singularity spectrum of the port towns. Similar calculations are possible for generalized dimension spectrums.

In order to understand the curve of these spectrums or their asymmetry, researchers have used other, occasionally more classical statistical tools. Thus, in some analyses symmetry is assessed from the asymmetry coefficient calculated on the values from a particular spectrum.



**Figure 6.9.** Multifractal scan of Barcelona, split into four sections

It is also possible to construct relative spectrums. These are obtained by subtracting the Euclidean dimensions of the phenomenon considered from the raw values. Thus we obtain the equivalent of a codimension. These relative spectrums enable the multifractality of objects that do not have the same Euclidean dimension to be compared, such as the multifractality of a line to that of an image. Meteorologists and hydrologists prefer to directly calculate codimension spectrums from cumulative probabilities.

Finally, it is possible to use multifractal scanning to determine the generalized correlation dimension or singularity spectrums from parts of images. It is also possible to observe how the form of these spectrums changes according to the subimage sizes.

Figure 6.9 represents the singularity spectrums of quarters of the Landsat image of Barcelona. We observe that multifractality varies very strongly; while it is particularly high for the north-west and south-east quarters, it is lower and more regular for the other two. It is possible to proceed to multifractal scanning on less expansive urban areas and to classify these spaces according to the amplitudes of their irregularities. Also, this type of analysis can be performed on any data, whether relating to buildings, population or social representations.

Another form of application has been proposed in comparative studies. Thus, in financial analyses, which mostly cover chronicle series, it is possible to calculate cross-correlations between spectrums in order to better define how irregularities correlate within dephased series. Such studies have been performed in order to couple the irregularities between the New York and Chinese stock exchanges. The results obtained are still the subject of some debate. Sometimes, simple correlations between spectrums have been calculated in order to analyze the relationships between the degrees of irregularity in a multiscale, but not temporal, context. It would even be possible to establish a classification of urban hierarchies according to their multifractal spectrums by using the algorithms described by P. Dumolard [DUM 11].

Finally, and above all, the scientific literature has flourished on cascades constituting applications of multifractal formalism. We recommend that anyone interested in understanding these techniques reads the work of L. Seuront [SEU 10], courses from the ENS in Lyon, and the recent works of meteorologists [LOV 10], who apply these tools as a matter of course. Let us briefly summarize them.

After Richardson's initial cascade, which represented turbulence in the form of whirlwinds within larger whirlwinds, specialists in this area identified the Mandelbrot and Barral cascades and the infinitely divisible Bacry and Chaisnais cascades. For the time being, these techniques remain confined to studies on turbulence. Schematically, using a Brownian model, the Hurst parameter is replaced with a series of values representing the results of multiplicative cascades. These phenomenological models, which are not therefore based on physical equations, should have the advantage of better representing climatic fields.

## 6.5. Conclusion

This chapter has examined the calculation of examples of multifractal formalism, since it is still being ignored by the geographical community. Certainly there is an educational interest in results obtained using experimental data. In more detailed research, it would be better to work with longer series, to make more comparisons and to increase the number of classifications on spectrums. In spite of everything,

these examples demonstrate the richness of multifractal formalism, even if the choice of treatment technique imposes methodological thinking and preliminary visual and statistical testing.

We do not presume to have covered this multifractal formalism completely. We have also favored figures over equations out of choice. Whatever the reticence of geographers, they have to accept that tools are at their disposal to tackle irregularities, discontinuities and spatiotemporal disparities in a multiscale context. They can therefore support their thought processes by using appropriate techniques.

Having reached the end of the first six chapters, the reader has a veritable palette of tools at his or her disposal to describe coarseness, irregularities and multiscalar and multiscale geographical disparities. We now need to explain these regularities or irregularity laws, and this is the aim of the next two chapters.



## Chapter 7

# Geographical Explanation of Fractal Forms and Dynamics

“That which has been believed by everyone, always and everywhere,  
has every chance of being false.”

Paul Valéry

Having emphasized the universality of fractal phenomena in geography, we should now explain them, since geographers – like all scientists – do not settle for descriptions. First of all, it is possible to make use of disciplinary explicative models, which geographers will be very familiar with. This is the subject of the current chapter. In Chapter 8 we will call upon more abstract, and therefore more general, models and theories that are outside the scope of geography alone.

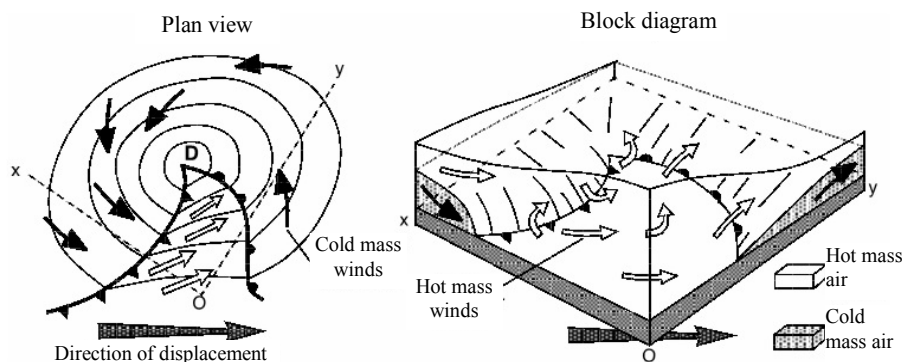
As our objective is academic, we do not need to consider every geographical phenomenon. We are only developing some particular examples, chosen based on our own fields of interest. By going into these examples in depth, we are trying to demonstrate that the fractal approach provides a new interpretation of phenomena that are considered to be relatively or even very well known by users of high school text books or first semester university works. We no longer need to explain mean tendencies, but how irregularities, differentiations and disparities are repeated or smoothed out in a multiscale context.

In physical geography, we take the example of temperate perturbations and then natural hazards, before more rapidly explaining some contributions from geomorphology and ecology. In human geography, we tackle the problems of population and populating, urban hierarchies and urban spread, before concluding on social interaction transformations, notably the influence of the internet.

### 7.1. Turbulence generates fractal perturbations and multifractal pluviometric fields

Even if some meteorologists have expressed doubts, in every high school text book or student climatology publication, temperate rainfall is attributed to the passage of Norwegian perturbations. The Norwegian perturbation is an abstract construction, a model that was proposed at the beginning of the 20<sup>th</sup> Century and explained the successive wet spells that hit the Scandinavian coastlines.

The Norwegian perturbation model is based on a deformation of the polar front, which separates the tropical warm air from the cold polar air (see Figure 7.1). An undulation is formed over this front, the warm air being trapped between the subsequent and earlier cold air. At ground level, the warm front overtakes the cold front, pushed by the polar flows. The subsequent cold front moves to catch up with the warm front, thus leading to the occlusion of the perturbation at ground level. In a vertical plane, the slightly angled warm front provides low precipitation, whereas the cold front, which is straighter, causes more substantial rainfall, before a sky dotted with lines of cumulus announces the return of beautiful weather. This Norwegian perturbation is often displaced in a westerly current with other perturbations.



**Figure 7.1.** *The Norwegian perturbation model*

This initial schema has increased in complexity. First, this model, which described the surface situation, was linked with flows circulating at altitude, notably jets and their cyclonic or anticyclonic undulations. Second, very soon afterwards secondary fronts were being identified, notably during the 1950s. In spite of these complementary contributions, the Norwegian perturbation model described in all high school text books created unanimity among researchers, if not in meteorology then at least in climatology. In P. Pédelaborde it had a staunch defender, since it was used by him as the basis for the classification of weather types.

In 1975, within the context of a thesis on precipitation in the Midi region of France, the use of complex statistical techniques that are now considered commonplace, such as multidimensional classification, and comments built around a theory of systems, led the Norwegian perturbation model to be called into question. Thanks to the use of satellite photography, J. Mounier confirmed that more than a quarter of rainfall levels recorded could not be attributed to the passage of a front.

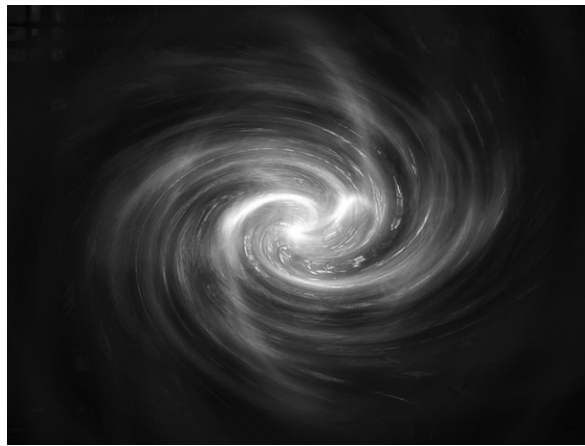
Without attaching excessive significance to field practice, if the Norwegian perturbation model was relevant, all graphs drawn using pluviographs would present two peaks corresponding to the passage of a warm front, then a cold front, except in the case of occlusion. This type of bimodal curve is, however, excessively rare. Pluviograph curves are much more irregular.

The Norwegian perturbation model was based on the idea that the atmosphere is regulated by linear distributions, yet it is now accepted that the atmosphere is an open dissipative system whose behavior is nonlinear.

Despite its continued daily appearance on TV, the Norwegian perturbation model is therefore no longer relevant [DAU 98]. Temperate domain perturbations, such as those in other climatic zones, are fractal perturbations. They are not comprised of two fronts majestically displacing each other from West to East, but consist of a multitude of fronts, rapidly appearing and disappearing in more or less direct relation with the conditions of altitude. In a spiral fractal perturbation, the divided fronts, which are broken down into smaller fronts, swirl around the heart of the depression.

More precisely, the new model proposed depicts a fractal spiral. It constitutes a complex assembly of curved fronts whose shape is repeated on different scales in accordance with the principle of auto-similarity, which is identified in every fractal. Its graphical representation (see Figure 7.2) is testament to the similarity between a fractal and a very hollow depression, with its resultant patchy cloud cover. However, in science a visual impression is not enough, and other proofs have to be constructed.

This model has the advantage of being more efficient than the Norwegian perturbation model in terms of taking account of the reality observed. First, it provides a good explanation of the spatiotemporal irregularity of rainfall recorded by pluviographs. The temporal and spatial irregularity of downpours is no longer abnormal, but corresponds to the multiplicity of fronts that are succeeded irregularly for a given area. What the very regular Norwegian perturbation model was not able to explain, the new fractal perturbation model was able to account for without any difficulty.



**Figure 7.2.** *The fractal perturbation model*

This new theoretical model also has the advantage of better accounting for the rapid succession of frontogenesis and frontolysis processes. In the Norwegian perturbation model, the fronts are displaced slowly, without really being altered, apart from at the start or end of life; and yet the reality is very different. The front is a place where ascending and descending forces are acting continuously. This permanence of the frontogenesis and frontolysis processes explains the incessant birth or regeneration of fronts, and the same goes for their disappearance.

Finally, and importantly, this model has the advantage of conforming to new theoretical concepts that have been proposed by meteorologists and geophysicists, who have emphasized the essential role of turbulence. Turbulence, which is always present in the atmosphere, is the physical process that governs the life of a depression and the associated perturbations. A turbulent flow is the result of winds that are trapped and of different ages. These vortices are arranged into threadlike spatial structures, observed in nature and in digital simulation [CHE 01]. In fact, fractals and turbulence are the two faces of the same reality: “For turbulence, geometry is subordinate to the dynamic, while for fractals it is geometry which

dominates. Fractals have given us a new language within which we can describe the form of turbulence” [STE 89]. In this quote, geographers find the classic dualism that is familiar to them, between process and form, and therefore the possibility of conducting their studies by moving one away from the other.

Additionally, we can no longer count on articles demonstrating that fractality is more generally the multifractality of pluviometric fields when the intensity and not just the presence or absence of rainfall is analyzed. B. Mandelbrot [MAN 75] was the first to attribute a fractal form to clouds. This hypothesis was verified by S. Lovejoy [LOV 82], even if the auto-similarity effect was not ideal over the complete range of spatial scales considered. In 1985, S. Lovejoy and B. Mandelbrot [LOV 85] proposed a model for taking account of the fractality of pluviometric fields observed using radar for Montreal, Spain and the tropical Atlantic. There have even been whole theses devoted to this subject [BIA 04, GRA 03]. Thus, the first of these researchers demonstrated the multifractal characteristic of pluvius formations revealed by infrared satellite photography. A summary text, which appeared in *La Houille Blanche* review in 2006, quoted numerous French works. Some similar conclusions have also been drawn on an international level [LOV 95], [SCH 87]. The multiplicity of pluvius fields is a general distribution that cannot be explained by a Norwegian perturbation, but for which a fractal perturbation is a perfect fit.

This demonstration is not applied to single perturbations from temperate zones; it also includes tropical perturbations: all perturbations are fractal. In a recent article, S. Verrier *et al.* [VER 10] demonstrated the multifractal characteristic of rainfall from the African monsoon, with maps produced from data from the RONSARD Doppler radar. Previously, P. Hubert [HUB 88] had proven the multifractal characteristic of the temporality of tropical rainfall, caused by grains, and analyzed a downpour in the Ouagadougou region in detail.

Thus we have found one of the main points of all new theoretical modeling: it causes common traits and barely perceptible convergences between earlier models considered distinct. Temperate and tropical pluviometric fields are both multifractals.

Finally, all meteorologists link multifractality to turbulence. We will revisit these turbulent motions, which animate natural and human flows, in Chapter 8. Far from the regularity of the Norwegian model, which is incapable of explaining the irregularity of pluviometric fields, the fractal perturbation model agrees with the “reality” that is observed.

## 7.2. The fractality of natural hazards and catastrophic impacts

Natural hazards, seismic events, volcanic eruptions and floods – on which a number of risks are assessed – follow some fractal distributions. These fractal distributions are initially applied to frequential intensity distributions for various hazards. The best known, the Gutenberg-Richter distribution, links the number of earthquakes with the energy released by them, and therefore determines their magnitudes. Its fractal interpretation, which was justified by K. Aki [AKI 81], suggests that small tremors and devastating earthquakes have the same origins, but this fractality is also signaled in river flow intensity curves.

Fractality also characterizes the territorial organization of natural hazards. Many studies have confirmed this for earthquakes. From this, an earthquake is no longer induced by a single fault but is dependent on the energy accumulated in a fault zone. The study of fault zones has become more relevant with our increased understanding of seismic phenomena. It no longer makes sense to analyze earthquakes in isolation. A summary of the most detailed works on this subject has been developed in the publication written by J. Dubois and J. Chaline [DUB 06].

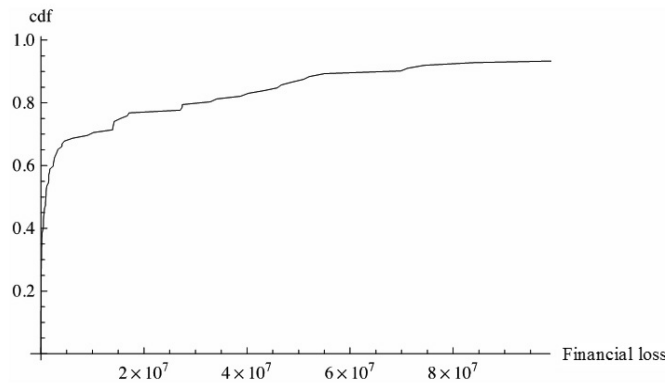
This fractality also characterizes volcanic lava flows. According to H. Gaonac'h [GAO 92], it is caused by rheological nonlinear distributions combined with gravity and the underlying topography that covers these flows. The explanation of these fractal morphologies still resides in the shape-generating effect of turbulence, but in a viscous environment, and the lava that flows down the slopes of volcanoes.

Finally, fractal distributions provide order to the temporal evolution of the majority of hazards. This fractal approach, which is applied to the temporal order of hazards, is a source of interpretations that go against the ideas that are currently accepted and promoted in most geography manuals. The adjustment of the time intervals between earthquakes using a power law suggests that the longer the time interval, the further away the next earthquake will be in terms of time. This distribution contradicts the catastrophic predictions and blockbuster movies that announce that imminent devastating earthquakes will be closer together. The power law concludes the contrary and tends to delay these exceptional catastrophes, which will certainly take place, but further into the future.

Above all, for a number of years, various researchers have made use of log-periodic distributions in order to modelize seismicity curves. This distribution superimposes an increasing or decreasing periodic rhythm onto a power law and enables the small seismic thrusts prior to a large earthquake to be taken into consideration or can be used to reproduce aftershocks.

In conclusion, the understandings drawn from the fractal approach are providing a new interpretation of these hazards. High school and often university manuals provide sufficient verification on this. In these manuals, hazards are usually interpreted as disorders or accidents, and yet the fractal explanation of these phenomena teaches us that the opposite is true. Disorder must be replaced by a learned order, which will certainly be complex at first glance, but will still be fractal. The spatiotemporal irregularity of hazards is regular, universal and subject to geophysical laws. This iterative order is the product of energetic interactions between spatiotemporal scales.

Moreover, still within the domain of risk, fractality is not applied to single natural hazards. It characterizes other biological and social hazards, and even more generally the vulnerability or resilience of risks and catastrophes.



**Figure 7.3.** *Pareto I adjustment of financial losses from major catastrophes*

For example, we had shown that the financial losses due to major catastrophes follow a power law [DAU 01]. For an updated data set, covering the period 1900–2010, as financial losses increase in millions of euros and are therefore strongly positive, all adjustment models provide good results when the determination coefficients, which are rarely below 0.98, are examined. This is similar to the previous assessment performed on large towns. However, the Pareto I model is always the best, according to the Akaike and Bayes tests. This model, which is illustrated in Figure 7.3, is also suitable for all catastrophes when taken together. We have verified this for earthquakes and floods from data collected by the Centre for Research on the Epidemiology of Disasters.

In order to explain this financial evolution of catastrophes, we must draw on formal theories, the complexity of which will be explained in Chapter 8. No

disciplinary theory manages to explain the fractal universality of the majority of risks, whether they are natural, technical or social.

### 7.3. Other explanations from fields of physical geography

While we have devoted some pages to climatology and risk, it should not be concluded from this that the fractal paradigm only slightly changes our representation of other geophysical phenomena. Indeed, a number of contributions exist in marine biology and ecology, but also in understanding the organization of soils and relief forms.

Relief forms are fractal, which is no surprise, since forces and erosion mechanisms that are principally represented by pluviometric intensity fields and river networks in temperate environments are also fractal. These forces are also applied to faulty tectonic structures. This compatibility between mechanisms and forces is not, therefore, unintentional.

In all known studies for a given region, the fractal dimension increases with altitude. This amplification of irregularity due to altitude is the result of growing erosion, which increases the topographical fractal. Classic geomorphologists know this already, but the fractal explanation offers the benefit of allowing a better understanding of the growing irregularity of processes and forms.

These forms of fractal relief are occasionally explained by coupling Culling's equation, which links the erosive flow of the gradient to a stochastic process, which assimilates the nature of rocks and rainfall to random noise. As this erosion model, which leads us to the Kardar-Parisi-Zhang equation, is not linear. The topography is therefore better described as a multifractal spectrum [SOR 00]. Indeed, simple fractal models are often additive, as multifractals are produced using multiplicative effects in cascades.

Such explanatory models are a departure from the classic practice of map comments, where reliefs are explained in terms of direct and linear relationships between topography and the tectonic on one hand, and topography and past and present climate on the other, even though the grand masters of geomorphology insisted on the multiplicative combination or causality, especially A. Cailleux.

Let us now go into a little more detail regarding coastal morphology, to which fractalists have devoted many pieces of work [BAK 99, OKU 01, SCH 89]. Certainly, coastal erosion generally increases the irregularity of the coastline, but this rise in coastal irregularity has in return led to greater erosion from sea winds. Crashing waves have led to a reduction in its erosive power. Thus, the combination

of these contrasting mechanisms has led to the emergence of a stationary fractal coastline [BAL 04]. As B. Sapoval explained [SAP 97], it is no longer a case of the sea attacking land, but the other way around. Through its geometry, the land is weakening the erosive impact of the sea.

More precise analyses suggest that a balance is attained when the fractal dimension of the coastline is equal to 1.33. By keeping this value, it is becoming possible to classify coastline according to its state relative to this dynamic equilibrium state. Thus, contrary to the idea that is generally accepted, linear coastlines are not necessarily protected from erosion. Instead, the opposite is true: it is fractality that protects the coastlines from erosion.

In order to take account of the fractality of other phenomena of physical geography, we recommend a number of specialized publications. Thus, for all continental hydrology, it is essential to refer to the treatise of I. Rodriguez-Iturbe [ROD 97] and to P. Martin's *Accreditation to Supervise Research* [MAR 04]. Certainly J. Philipps [PHI 93] has already revealed that river networks for which the fractal dimension was strong were not developed on constraint geological structures. Such river networks tended to completely cover recently sedimentary sites that were not in conflict with the diffused water flow. This is an application of Brownian or diffusive motion. These initial studies, however, received too little prominence.

Studies conducted by L. Seuront constitute an introduction into fractal analysis in the domains of oceanography and marine ecology. As in meteorology, the emphasis is on the turbulence of marine flows that generate multifractal forms. The Gulf Stream and most of the major currents are turbulent, as shown by satellite photographs. Phytoplankton is led by this turbulence, which partially explains the displacement of micro-organisms following a Lévy flight, with short displacements at the heart of the vortices, intersected by long-distance drifts.

Out of all of these examples that are tackled in physical geography, let us keep the impact of turbulence in mind. This is present in the atmosphere, hydrosphere and also the lithosphere. This mechanism is generating fractal forms and multifractal intensity fields throughout; this is true of pluviometric fields, but also of the Gulf Stream. The explanation of fractal forms and dynamics does not only concern the "natural" or biological world, however; man-made forms are also fractals.

#### **7.4. A new geography of populations**

High school, but also university geography, text books examine the distribution of populations based on an average density calculated over the whole world. Their authors highlight disparities using deviations from this average density as a basis.

They provide a disordered view of population, which they occasionally attribute to man's freedom of movement.

This would make sense if the distribution of densities was Gaussian, but this is clearly not the case and an interpretation relative to the mean is hardly logical. Areas with differing levels of population densities, whether they are town centers or peripheries, are observed on all scales – from the global to the urban or even a simple village. Although it is not very realistic when analyzed for a single level, the center/periphery model becomes credible in a fractal and multiscale context. It is thus similar to an iterative mechanism, which is a source of fragmentation and population irregularities. Beyond the apparent disorder, population therefore obeys an order of scale.

The visual examination of rank-size graphs is certainly not enough in itself for a precise choice of population range distribution, but it does show that Gaussian distribution is not suitable. The distribution of populations following a State census follows a simple or general Pareto distribution, or even a parabolic fractal. The average density has no relevance here. Another approach, using maps or night-time satellite photographs, confirms that the distribution is non-Gaussian. Strong densities appear clearly and impact on the mean, and yet all calculations performed on these photographs attest to a fractal population distribution.

Geographers and demographers have advanced a number of explanations in order to account for these fractalities. In his thesis, M. Forriez [FOR 10] calculated the fractal dimension of the populations for each continent. He compared Eurasia, which is far too highly populated, to the Americas, which are rather more recently populated and whose fractal dimension is clearly lower. Forriez draws on the history and formation of the first settlements in order to explain the differential fractality of a population.

For H. Le Bras [LEB 00], however, this multifractal population distribution is based on the conflict, identified by Kant, between the desire for association, for living as a group, and every person's desire for isolation. The explanation is therefore in the individual behaviors of each person. Without wishing to deny the interest in this type of interpretation, the link between a multifractal – which H. Le Bras is confusing with a fractal dimension that varies according to continent – and both of these mechanisms, remains unclear. The interaction between individual but generalized behaviors and worldwide distributions, which are indispensable for explaining fractality, still includes some gray areas.

Other demographers attribute this multifractality to natural demographic growth, coupled with a local concentration effect. They have constructed some simulation models showing the minimal impact of other factors, such as differences in soil

fertility or the attraction of towns. They have not radically modified population multifractality. These criteria do, however, have an impact on the precise localization of people and deserted spaces.

Other, still more general, explanations will be presented in Chapter 8.

Contrary to what is revealed in manuals, however, geographers have to explain population multifractality, which relegates the physical and economic factors to the second level, after human behavior and the weight of history that is interacting at all levels in time and space.

### **7.5. Harmonization of town growth distributions**

The growth of towns, notably in terms of people and buildings, has given rise to an abundance of literature. Geographers are faced with three distributions, which appear contradictory at first glance. The densities of urban populations were first measured using the Clarck model, i.e. an exponential decrease in distribution from urban centers to their outskirts.

This model, which has often been verified, was applied to diverse variables, for example property prices or the density of buildings. This distribution remains the subject of multiple interpretations. For example, for various authors, it indicates an equilibrium state for a diffusion process [BAT 92, BEA 97]. Sometimes it results in short-distance spatial interaction in certain multiagent simulation models. The new urban economy, which picks up on Von Thünen's principle, accentuates the credit of an areolar configuration for a monocentric town, conforming to this principle of exponential decrease from the center to the periphery.

Several articles by M. Batty and P. Frankhauser [BAT 95, FRA 94], which have inspired a number of studies on very different towns in Europe and Asia, have confirmed the fractal characteristic of this decrease. According to all of the authors of these studies, buildings are organized according to a power law. From all the evidence, this fractal dimension varies by town quarter, as well as indirectly varying from the center to the periphery. It does, however, remain the signature of a power law.

Finally, when towns are described according to a measurement – for example population density or some other measurable criteria, such as the percentage of social categories – and therefore by an image or grayscale map, some authors have demonstrated the multifractal characteristic of these urban fields [BAT 94, CHE 07]. In Chapter 6, we verified this multifractality for the great port towns of the Northern

Mediterranean coastline, which were depicted by Landsat 7 photographs. It is therefore no doubt reasonable to accept this rule for all great agglomerations.

Any geographical explanation must take account of the coexistence of these three distributions. There are, no doubt, some mathematical links between them. Thus, a power law may be the product of two exponential functions, but, to our knowledge, there is no complete and satisfactory disciplinary explanation for the expansion of a town.

Multiagent models – which simulate the behavior of residents according to the cultural logic explained by T. Schelling [SHE 80] – produce fractal or even multifractal segregation. At least we can assume this, since multifractal formalism is rarely introduced in order to characterize territorial fields obtained through these simulations. We have, however, verified this for configurations produced by simplified models, which have been elaborated by R.J. Gaylord and L.J. D'Andria [GAY 98]. However, other ingredients must be introduced in order to arrange this fractal social segregation into rings reproducing the Clark model, and then to localize the rich, poor and mixed areas.

By way of working assumption, it is possible to put forward that fractality is the result of cultural segregation, and therefore of a more or less normal diffusion mechanism, while center-to-periphery ordinance falls within economic mechanisms, which are better explained by the new urban economy. This type of explanation remains imperfect and recourse to formal theories on complexity has become a method followed by various geographers and economists. These theories are tackled in Chapter 8.

In spite of knowledge gaps during the course of elaboration, these types of explanations are rarely set out, as far as hypotheses are concerned, in the great texts on urban geography. Added to this, they are completely ignored in high school education.

## **7.6. Development and urban hierarchies**

For a long time, the urban hierarchy formalized by Zipf's law was observed, but not explained. Geographers classified urban networks by comparing them with this distribution. They also emphasized the history of particular towns, such as Paris, in order to take account of their differences relative to the distribution, whose foundations remained unexplained. In previous chapters we have noted that the best adjustment, and therefore the most suitable distribution, is certainly dependent on the data used, but often remains a Pareto I or IV model.

Schematically, it is possible to assert that the set of all human, urban and rural communities is not ordered according to a Pareto IV distribution as often as a log-normal distribution. However, by analyzing single towns the Pareto adjustment is more effective. Finally, by eliminating the small towns and concentrating on the large ones, a Zipf distribution is often acceptable. Indeed, the more small human concentrations we eliminate, the more the hierarchy tends to be ordered according to a Zipf distribution. Thus, the 30 biggest towns in the United States are ordered according to a Zipf model, but the Pareto model is better when the number of towns to be taken into account exceeds this threshold. The rank-size distribution of urban hierarchies is not a uniform law of scale, but simply an asymptotic law.

Additionally, although it is universal, the gradient of the adjustment appears to change over time. M. Dimou *et al.* [DIM 08] provided a very good demonstration of this for Chinese towns. From 1960 to 2000, the Pareto coefficient did not stop increasing, which indicated a more significant weighting for average towns compared with great metropolises. This scientific assessment of an accelerated progression of average towns is contrary to the message promoted by the media, which have increased reporting on individual large metropolises based on their accelerated growth. In fact, China is not a countryside eroded by several gigantic cities, but a country that has been profoundly urbanized at all levels.

Economists have drawn similar conclusions when analyzing the hierarchy of businesses. Based on the data given, the distribution that governs them is the same. This change in distribution can without doubt be applied to other phenomena. While the Pareto-Zipf split is not of major importance, the distinction between a log-normal and Pareto adjustment is important, since we pass from the ordinary, non-auto-similar world to the fractal one.

For Zipf, the explanation of this order was the principle of least action. The structuring of language was, for him, optimal, relating to two contrasting forces, described as unification and diversification. This finalist-inspired principle has been strongly devalued. Although it is now generalized under the heading “variational principle”, the least action principle has become essential within quantum physics.

In order to avoid false debates, it would be advisable to separate finality from intentionality. All physical or human systems have at the very least one finality to be maintained. Conversely, physical systems have no intentionality. A stone does not intend to fall and injure a passer-by. Reconsidering the principle of least action by establishing this distinction would suggest that the rank-size organization of towns is a result of an optimization process. This optimization principle remains to be discovered.

Some authors have tried to establish a relationship between this urban hierarchy and certain economic variables, particularly the gross domestic product. Most of these authors are economists who are almost blindly followed by geographers. In geography, B. Berry [BER 64] successfully tested this hypothesis, but this type of explanation is not shared by the whole scientific community. More recently, a number of authors have linked the distribution of towns, not to an economic state, but to growth and therefore to a dynamic.

For some, notably H. Simon [SIM 55] and J.C. Cordoba [COR 08], this growth is random in nature. The growth of towns depends on exogenous shocks. The dynamic of a town follows a Gibrat distribution, otherwise known as a log-normal distribution. More precisely, the stable state of an urban network or of a set of businesses obeying a Gibrat principle would be described by a Zipf distribution [GAB 99]. Similar growth at all scales, for all classes of businesses and all towns, would therefore explain the scale invariance of the urban network. The demonstration is irrefutable and the attraction of this explanation for some geographers is understandable.

This choice, however, suggests that urban growth is independent of town size, that it is non-auto-correlated in time, and that the towns are not interacting. These hypotheses are strongly reminiscent of those applied to an ideal gas. In this physical model, the molecules are not interacting. However, in 1873 J. Van der Waals abandoned this model, and then studies on magnetism definitively condemned them. Despite these hypotheses being condemned, economists are preserving this hypothesis of an absence of interaction between the towns in an urban system, which is quite difficult to accept. Various authors have therefore proposed completing H. Simon's model by taking account of, for example, the cost of inter-urban transport, which comes back to the interaction between towns.

It should be noted that when a town loses its ranking, it does not get it back systematically, and the hierarchy is therefore permanently modified. The study undertaken by M. Bosker *et al.* [BOS 08] aimed to include this by using the evolution of the rank-size distribution of German towns bombed during the World War I and II as an example.

A further issue with the choice of distribution is that Gibrat's law ignores the emergence of new urban settlements. This assumes that no market towns or villages develop or are transformed into towns. This too simplistic hypothesis reinforces the doubts about this type of explanation.

Despite all of these difficult-to-accept simplifications for a network of towns, this model was considered pertinent by various specialists in urban geography. F. Moriconi-Hébrard [MOR 98], who took account of all these criticisms and some

statistical tests – which were too hurriedly put together – rose up with good reason against the confidence attached by his colleague D. Pumain [PUM 82] to the Gibrat theory. He was also opposed to any deductive, and therefore theoretical, enterprise.

Conversely, for supporters of endogenous growth, the impact of size on town growth is very real. Growth is based on innovation, which is dependent on the size of agglomerations. Innovating businesses are grouped in the large metropolises. These explanations refer to Schumpeter's ideas. With such endogenous growth, the larger towns must preserve their ranking; they cannot be declassified. However, the results of practical tests on different national urban hierarchies are not all conclusive.

This is why certain models, such as the Duranton model [DUR 02], combine these two types of growth – endogenous and random. The localization of innovation is no longer determined by the size of large metropolises, but is supposedly random. This model offers the advantage of including a source of differentiation between towns. Useful reading for anyone striving to go into greater depth on this issue should refer to A. Schaffar's summary article [SCH 09].

More recently, D. Pumain [PUM 06] has proposed a theory, called *changes in urban hierarchies*, which makes use of Schumpeter's ideas. Having observed that, according to the criteria retained, the exponent of the power law is less than, equal to, or greater than 1, it links these laws of scale to cycles of economic innovation and hierarchical diffusion. For example, for the most innovative economic sectors, the scaling parameter is always above 1. Conversely, it is less than 1 for end-cycle economic sectors. This intellectual construction assumes that the scaling laws are linear in nature, which we still need to demonstrate, especially as they are not log-normal, which appears to already be accepted by a number of geographers.

All of this reasoning is not enough. On one hand, it badly explains why we find not only the same distribution but the same parameters in urban hierarchies as diverse as those in India and the United States, and also why identical parameters pertain not only to urban hierarchies but also a number of variables that define the World Wide Web and the corpora of languages. This universality has forced a transdisciplinary approach. On the other hand, these explanations do not take account of the spatial fractality of the urban network. Even if deformations are always perceptible relative to W. Christaller's deterministic hexagonal fractalities, urban networks are well ordered in space as a result of a fractal structure. A satisfactory theory must simultaneously explain frequential fractality, and therefore the rank-size distribution and the spatial fractality of the distribution of towns.

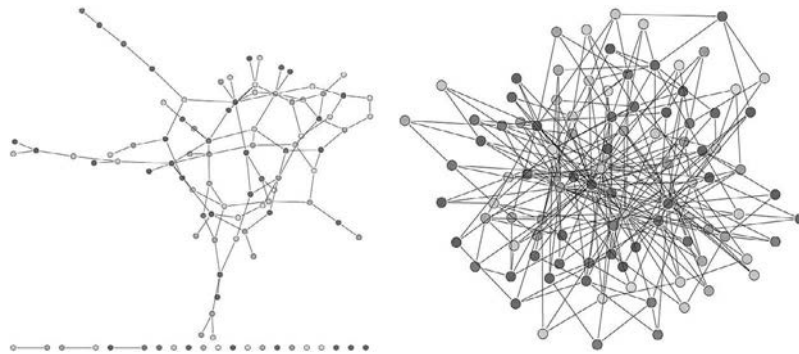
### 7.7. Understanding the formation of communication and social networks

The internet has profoundly changed the lives of modern-day societies, as it has altered the social link between individuals. A number of researchers have therefore focused their attentions on the development of this technological innovation, which has imposed its logic on numerous stakeholders and their activities. These studies have applied to the topology and geography of this network and on the organization of social communities that have latched on to it.

Represented in graph form, the internet has three properties:

- It is a “small world” set, i.e. a sparsely populated graph, with fairly small distances between the nodes of the network.
- It is formed from clusters that represent the existence of numerous communities, such as scientists, travel enthusiasts, gardeners or fine wine buffs.
- It is scale invariant, as proven by the existence of many power laws.

In order to explain the characteristics observed in many other networks, various theories have been put forward [PAS 04]. Static random graph models are no longer topical; rapid evolution network models are favored over them. The Barabasi-Albert theory has put forward the mechanism of preferential attachment, where points that have already been frequently visited receive even more visitors (see Figure 7.4). A micromechanism of growth causes nodes to emerge that are arranged according to a power law.



**Figure 7.4.** Random graph and a Barabasi-Albert graph  
(source: Mathematica)

This theory can be illustrated by a simplified simulation model, where a new line is created at each time step, and these new lines are linked to already-existing points

according to a probability that takes account of the degree of attractivity between each of them. This theory provides a good explanation of the small-world effect, the cluster effect and scale invariance.

This theory does not explain the emergence of new points, which suddenly become highly attractive and provoke a reorganization of the network, as shown in the real-life example of the rapid and dramatic rise in the popularity of Google. This is why, in more recent theories, a supplementary distribution expresses the tendency of certain points to more rapidly attract the newly created points. This law is a transposition of the mechanism of natural selection which has been adapted to evolution. It no longer applies to single living species, but to the emergence of a technical or social network.

When the network includes hubs, i.e. preponderant or dominant nodes that are frequently encountered, these hubs are sometimes repelled by each other. This effect is traditionally considered to be the result of competition. This mechanism is often described in aerial transport networks [GUI 07, HU 09], where competition between the major airports is very lively. This effect, which is in contrast to preferential attachment, is without doubt one cause of the break observed on a straight-line adjustment graph, which is also due to the fact that most of these networks are characterized by semifractals or a double or truncated Pareto distribution.

Thus, the modern world, which is increasingly transfigured by the dynamics of communication and information networks, is a world where fractality is increased. Globalization is accompanied by still greater fragmentation. We better understand the feeling of disorder, which appears to be widespread. This disorder is in fact superficial. Citizens of the new world live in multiscale environments created by their own interactions. Yesterday, the world was dominated by diffusion, Brownian motions. Today, we are entering the world of Lévy, a world of turbulence.

## 7.8. Conclusion

There are three conclusions to be drawn from these few examples of disciplinary explanations. First, the fractal approach is no longer solely confined to the description of phenomena, notably irregular non-Euclidean forms. It is forcing new explanations.

Secondly, these explanations are also profoundly renewing the vision of all geographical phenomena. All of the examples presented in this chapter demonstrate this. These explanations of scale invariance should stress the interactions between levels, not separate them. Whatever its merits, the Gibrat model, which is often invoked and generalized through the introduction of thresholds, but ignores the

interactions between towns from the same level, attests well to the effects of this incompleteness. This priority, which is accorded to interactions between levels, must also be applied to other logic, such as earthquakes and population.

Thirdly, most of these disciplinary explanations remain incomplete. Based purely on these explanations, we have a poor understanding of why totally different objects obey the same power laws. We must therefore return to complexity theory, which offers the advantage of bypassing this disciplinary context, and is the subject of the next chapter.

## Chapter 8

# Using Complexity Theory to Explain a Fractal World

“Although specialization is a necessary trait in our civilization, it should be completed by the integration of a thought which crosses disciplines. A permanent obstacle to this integration is the demarcation line between those for whom the use of mathematics is easy, and the rest.”

Murray Gell-Mann

The fractality of morphologies and morphogenesis does not stop at the gates of geography. It relates to all disciplines, all objects in our universe and the organization of things such as DNA sequences and the distribution of galaxies across the universe. This universality of power laws has led to more general explanations being proposed that are valid for both physical and social sciences.

In spite of everything, this logical deduction, which we have shared with numerous scientists, offends all those who defend disciplinary knowledge. It provokes reservation, even outcry, which deserves attention if not resolution in the first part of this chapter – even if we do not hope to convince disciples of human free will and to end this discord. After this outline discussion, section 8.2 is devoted to the examination of general mechanisms that generate the power laws.

Some formal, very general theories are then detailed in section 8.3. The advantage of these is that they explain the fractality of a vegetation map or a social network built around an internet addiction, a forest fire or urban spread. Four theories respond to this need for universality, the theories of P. Bak, A. Béjan, L. Nottale, and one theory that we presented at the *Géopoint* symposium in 2010. Now we are exploring a number of mechanisms for generating power laws, which are the probabilistic representations of fractals. Fractals are powerful tools for understanding chaotic systems. We will try to show this in section 8.3.5.

### 8.1. A bottomless pit debate

Many geographers and specialists in social science are critical of complexity theories, their physical origins and their contempt for human action. They forget the richness of reflected or irrational behavior of the agents who have put together the economic or geographical spaces. This argument, which has been fairly confidently put forward, is without a doubt older than the social sciences themselves. It therefore merits being tackled, but without the claim that it is able to resolve the debate, since even a substantial study would not be sufficient for this.

On a philosophical level more particular in ontology, we understand the position taken by idealistic thinkers – the disciples of Plato. As they are dualists, they separate thought from matter, and therefore the physical from the social and cultural. For them, the distinction between physical and social sciences has a plausible foundation. This differentiation is less justified for materialistic thinkers, who affirm world ontological unity. Within such a single-tier philosophy, subject and spirit are interdependent and based on the fact that disciplinary barriers are useful, but arbitrary.

We must now depart from philosophers' territory and return to the scientific arena. Scientific history teaches us that the transgression of disciplinary barriers is useful and promotes progress. The first, most relevant example for geographers is that of Wegener, a meteorologist who anticipated plate tectonics by affirming continental drift. Geologists, outraged by this intrusion coming from a stranger to their discipline, tested his belief in successive seminars where they pronounced increasingly severe judgments on his theory. It can of course be argued that this was an internal issue within the same area of knowledge – the geosciences.

Another symbolic example bypassing the context of this book relates to three great thinkers of the 19<sup>th</sup> Century: Malthus, Darwin and Marx. The Malthus principle – the fight for life – became Darwin's principle of natural selection. This therefore passes from the economic sciences to the life sciences, before moving to political science under the heading of Marx's class struggle. The last two thinkers,

Darwin and Marx, clearly indicated that they followed the Malthusian principle and transferred it to their own discipline. Naturally, numerous scientists have highlighted the limitations of these theories. Without doubt, however, they contain an element of truth, except for hard-line creationists or disciples of an exclusive methodological individualism, who attach no values to social categories and classes.

It would be all too easy to show how chemical reaction–diffusion, ecological predator–prey and exploited–exploiter models, which are all similar, circulate freely between the different sciences and are mutually enriched as a result. They were first formalized in biology, before transferring to chemistry, followed by diffusion into economics.

Additionally, there is nothing to prevent the introduction of human behavior into the models that were initially developed in the physical sciences. In different chapters of this book we have seen how Brownian motion, which is synonymous with normal diffusion, is complexified by constraints that are occasionally of a physical nature, but may also be representative of the reflected action of man. Similarly, we have identified a narrow connection between the Gibrat and ideal gas models. Both of them ignore the interactions between molecules or towns. Indeed, the physical sciences included interactions in their models well before the social sciences did.

This acknowledgment came about through the use of macromodels developed using differential equations, as brilliantly illustrated by the works of H. Meinhardt [MEI 95], or through the use of cellular automaton micromodels or multiagent systems. The innumerable macro- or micromodels, which represent diffusions encountered across all the sciences, demonstrate this. Moreover, multiagent systems, like their precursors the cellular automata, are rooted in statistical mechanics.

There are advantages and disadvantages to all types of modeling, which are tackled in various works [GUE 05, LAN 10, SAN 01]. However, successive contributions to simple models, invoked in the name of the complexity of living beings or social behaviors, must be mastered and properly understood in order to avoid unnecessary complexity, which is a source of incomprehension. We cannot help but give advice to young researchers: begin by establishing simple models then, little-by-little, adjust the supplementary rules in order to include multiple facets of human activity. The inverse approach often leads to an impasse, especially if these rules result in nonlinearity. Sometimes the results of simulation mask compensatory and artificial effects.

The constraint of the under-determination of models through observation is a formidable one. Effectively, as we are reminded by H. Atlan [ATL 11]: “For the same number of units, numerous connection structures with very different dynamics

can generate the same stable states". In concrete terms, for a multiagent system comprising only five agents, subject to some unique binary activity,  $2^5$ , or 32, states can be observed. These 32 states may be products of  $2^{25}$ , or around 10 million possible structures. It is therefore very difficult to attribute a state to a precise configuration of interactions. This is without doubt the main weakness of micromodeling, which is rarely evoked in geographical research, which favors this formalism. This constraint is a limitation of all micromodeling with the aim of being explanatory. It disappears when the objective is no longer understood, but only predicted, since the understanding of states is therefore sufficient.

All rational scientists agree that human activity is constrained by physical and biological metalogic and distributions. No human being can escape gravity. Allometry laws applied to trees or animals attest to this. So why do people subjected to these laws free themselves from social distributions? Unless they are claiming that there is no social distribution, which comes down to denying social-cultural segregation in all urban communities and refuting the existence of social sciences, we do not know the name of any principle whereby man would be free of any social constraints.

Conversely, man acts much more in the context of physical, living and social laws than gas atoms, which only confirm to physical laws. In order to understand the organization of a geographical space, none of these laws can be excluded. Consequently, for the author of this book, methodological individualism is expressed within a context. Through actions and interactions, man generates social and cultural laws that retroact on the action of each agent.

In order to progress in understanding the world, in physical as well as social sciences, levels must therefore be integrated and not opposed. Macro and microeconomics interact. Methodological and sociological individualism of categories and classes are also interdependent. They are invalidated when they are considered separately, yet fractal and multifractal formalism enables this multiscalar and multiscale integration.

Finally, the initial conditions – which are often considered to be random or homogeneous in physics – impose true preliminary structures that guide human activity. Man and society are subject to and constrained by this physical living and social metalogic. Nice, for example, is a town whose development is impeded and at the same time guided by the presence of the sea. The whole organization of its transport network, with traffic being concentrated on the axes parallel to the sea front, is dependent on it. Yet, acceptance of the power of metalogic leads to the acceptance of general mechanisms and transdisciplinary theories to explain world fractality.

## 8.2. General mechanisms for explaining power laws

Previous chapters have shown that similar power laws were discovered in very different fields of knowledge: in physics, seismology, economics and linguistics. Some authors have therefore advanced some general principles in order to explain these power laws, such as the rank-size distribution. According to M. Mitzenmacher [MIT 04], it is possible to identify three essential distributions.

The first of these is that of *preferential attachment*, which has been brought to light in recent studies on the growth of the internet. New interwoven links tend to be concentrated not by chance, but on the points most frequently accessed. Therefore, the most significant network nodes – described as hubs – harness the new access with a greater probability than nodes of lesser importance. To coin a phrase, “the rich become richer”, which has been described as the Jackson effect in economics and the Matthew effect in sociology.

This model proposed by A. Barabasi and R. Albert [BAR 99] is, in fact, equivalent to W. J. Yule’s principle [YUL 22], which was discovered in biology in 1922. Yule affirmed that mutations provoke the appearance of preferential new species within the existing category of the mutant species, rather than provoking the emergence of a new category. This micromechanism of preferential attachment, which has been iterated multiple times, provides a complex, scale-invariant network.

This principle was later taken up in economics by H. A. Simon [SIM 55] to account for the rank-size distribution of businesses. More recently, P. Krugman [KRU 96] emphasized the pertinence of this mechanism in the explanation of the rank-size distribution of towns, while observing that this contradicts the economy of scale principle, which underpins Lösch’s central place model.

New constraints, which have emerged in other fields of knowledge, are leading to the completion of this mechanism using new rules. They were well explained in F. Zaidi’s thesis [ZAI 10], which compared this mechanism to other models generating fractal social networks. He then proposed the integration of social regularities, for example the principle of homophilia. However, these supplementary demands do not challenge the scale invariance of simulated networks. Thus, heuristic optimal topology is a new model among many others that are widening the field of application of the preferential attachment model.

The second mechanism, pronounced and demonstrated by B. Mandelbrot [MAN 97], derives power laws from an information optimization principle, the mathematical formalism for which falls outside the scope of this book. However, the relationship between the Shannon information and the power laws is intuitively perceived, since both make use of logarithms.

This same connection was established by L. Seuront [SEU 10] for the single Zipf law. This *energetic optimization principle* also links the information and energetic entropies, even though they are not synonymous. This link is thus invoked by I. Rodriguez-Iturbe [ITU 97] to explain river network organization. This optimization mechanism is still the basis of constructal theory, which is explained below.

Finally, *multiplicative processes* that classically generate log-normal distributions may in certain cases be the basis of power laws. More precisely, it is accepted that multiplicative cascade processes create multifractals, while these same processes, simply added, form the basis of monofractals. There is a vast amount of literature on this subject and it is particularly abundant in all studies on turbulence, in physics as well as in meteorology [ABR 97, LOV 10, SCH 97]. Cascades and iterations are completed in order to produce fractality.

These principles, considered in isolation, do not manage to provide a clear understanding of phenomena, and even less so of forms considered. The possibility of generating scaling distributions is certainly practical, but this is not a fair explanation. It is therefore necessary to call upon more complete formal theories, namely complexity theories.

Contrary to what appears to indicate this adjective, these theories depart from simple considerations or laws in order to generate the complex, preferably by amplified iteration and mutation. This is not a question of celebrating the complex, but of defining how – based on the recurrence of simple laws – complexity emerges. The aim of science is to understand the complex, not to glorify it.

### 8.3. Four theories on fractal universality

There are many theories that lead to fractal organization. Four of these appear to us to be more convincing than others, even if their authors have tended to exaggerate their scope and the number of questions to which they can provide a response. We have therefore kept P. Bak's critical self-organization theory, A. Bejan's constructal theory, L. Nottale's scale relativity theory and finally a general theory of form that generalizes Turing's reaction–diffusion models, which we presented at the *Géopoint* symposium in 2010.

#### 8.3.1. Critical self-organization theory

*Critical self-organization theory* was proposed by P. Bak [BAK 87], but should not be confused with the many theories on self-organization. This theory is built

around two concepts: self-organization; and criticality or criticality [PRO 08]. Beyond these two principles, it is directly linked to the fractality of phenomena.

#### 8.3.1.1. *A theory built around two principles*

*Self-organization* is a property attributed to a system whose system is not the result of external forces, but of interactions between its own elements. A self-organized system is not subject to any external constraints from its environment, although it is open within this environment. These systems are robust or resilient, although they do not have any organizing center. Furthermore, self-organized systems behave nonlinearly: a small cause may produce great effects and breaks in the evolution of the system.

Long-range correlations between the components of the system appear in time and space, even if these components are also governed by local distributions through the interactions between closed elements. For example, in a convective cell, short-range correlations exist due to the interactions between neighboring molecules. However, a long-range correlation exists between an ascending current molecule and a molecule positioned on a descending current. When one goes up, the other goes down.

Self-organized systems are now being studied across all sciences. A wide variety of specific theories have resulted from them and so they carry different names:

- The dissipative structure theory, elaborated by I. Prigogine [PRI 77], places the emphasis on the constant energy feed required to maintain a structured system. P. Allen and M. Sanglier [ALL 81] transferred this theoretical model into geography in order to explain the formation of urban networks. They managed to bring about the emergence of new towns in a fixed entry network that evolved based on the number of jobs offered and the number of residents.

- Haken's synergetic theory [HAK 83] emphasizes two temporalities, slow and rapid, and the effects of self-regulation. His theory was successfully transferred into geography by L. Sanders [SAN 92]. These self-organized systems theories acted as a guideline for the works of D. Pumain *et al.* [PUM 89] in understanding the complexity of towns.

Despite this, spatial forms are of little interest in self-organized systems theory. They effectively couple simple differential equations where space does not appear. Critical self-organized systems theory, however, is a source of information on the spatial organization of a system.

*Criticality*, which is the second pillar of P. Bak's theory, characterizes phase-change systems, such as when water turns into ice. In this case, all of the elements of the system appear to influence the behavior of their close neighbors, as well as the

evolution of other elements further away. Based on this, the correlation between different points in the system is infinite. In reality, the system becomes critical when all of its elements are mutually influential. The critical point is the instant of dramatic behavioral change in the system, when it moves away from one attraction area in order to direct itself towards another attractor. From liquid, water is transformed into a solid (ice).

In social sciences, a majority vote is a good example of a critical system, all be it a political one. If two candidates obtain 30% and 70% of the vote, respectively, nothing happens when a few voters change their behavior. On the other hand, if the electorate is distributed 49.95% and 50.05%, a slight internal fluctuation induced by the displacement of a few votes is enough for a swing in the majority and the nomination of a new winner.

From this fact, critical self-organized systems are subject to a double evolution: a very slow general evolution, interrupted by rapid crises. Earthquakes, which violently free energy accumulated over the centuries, are an example that combines these two temporalities. This combination of two temporalities is also observed in urban networks. Very stable over a long period, such as a century, they are sometimes reorganized over a few years or decades. Without a doubt we are assisting this type of reorganization via globalization, with the resulting emergence of global villages.

These urban network reorganizations are not always self-organized and a high number of them are external in nature. For example, the industrial revolution changed the organization of urban networks through the introduction of new towns, but we are not referring here to a critical self-organization mechanism. This reorganization was the result of external perturbations, from the industrial revolution and not from internal fluctuations in the system of pre-industrial towns.

This is why physicists distinguish between different types of criticality. Self-organized criticality implies retroactions, while constitutive criticality corresponds to abnormal diffusion over a fractal support [LAG 03]. Geographers are faced with these different types of criticality, which merit a better differentiation in geography.

#### 8.3.1.2. *The double fractal signature of critical self-organized systems*

*Critical self-organized systems* pose a double fractal or multifractal signature. First, the variation between their states follows power laws. These first categories of distributions link the frequency or rank to the size or intensity of phenomena. The rank-size distribution of urban hierarchies symbolizes this first type. Second, other power laws characterize the temporal distribution of events. The distribution of changes in the share price of cotton illustrates this second type of power law.

Furthermore, critical self-organized systems construct fractal forms. According to P. Bak, only critical self-organized systems produce such forms, but this opinion is not universally shared. Beyond this uncertainty, critical self-organized systems theory therefore has an essential spatial component; it is a source of fractality, not only in time but also in space. Whereas some self-organized systems theories remain silent on territorial organization form, critical self-organized systems take account of space.

It is therefore possible to reverse the reasoning and accept that the system with a double fractal signature is consistent with critical self-organized system theory. Therefore we must not forget that two signatures are necessary in order to verify the theory, i.e. identifying a single fractal distribution is not sufficient. Our lungs fit a fractal form well, but they function quite happily without any critical fluctuation.

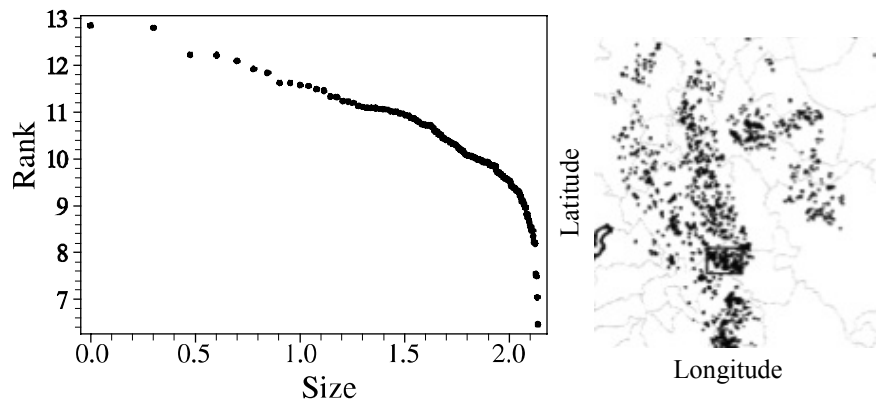
In previous works, we have put this theory forward in order to explain two geographically different phenomena: so-called natural risk [DAU 03a]; and urban networks [DAU 03c]. This metatheory was successfully applied to explain the evolution of a number of natural risks and disasters. It accounted for avalanches, notably powder avalanches. This metatheory has also been the subject of applications used to predict earthquakes. The interest in this is in the fact that it generalizes, rather than challenges previous explanatory models, notably the *Reid* and *spring mass* models.

Critical self-organization theory has also been used in studies into the evolution of forest fires, certain aspects of volcanic eruptions and even the Tokaimura nuclear disaster. It has even been used in the explanatory context of landslides, whose speed and size are linked to a power law (see Figure 8.1). In fact, it would appear that the Earth's crust is in a particularly critical condition, explained as much by natural seismic activity as by the microseismicity observed during the first filling of hydropower dams.

More recently, P. Bak [BAK 99] assessed that "The Cambrian explosion 500 million years ago and the Permian extinction of nearly 96 % of species 250 million years ago were the greatest avalanches ever to have taken place". This new interpretation, if it is correct, takes its place beside theories that draw on external mechanisms, whether they might be the fall of an asteroid or mighty volcanic lava flows in Siberia or on the Indian subcontinent.

We have also shown that this theory can be applied to urban networks (see Figure 8.2). We have known about the double fractal dimension of urban networks for a long time. However, no one in French geography had established this link between the fractal distributions of urban networks and critical self-organization systems theory. The towns in an urban network are ordered according to the first

fractal distribution: the rank-size, the Auerbach-Zipf or generalized Pareto distribution. Numerous studies have proved this.



**Figure 8.1.** *The two fractal signatures of landslides*

The hexagonal design of towns, which comes from central place theory, is a very good example of fractal territorial organization. In W. Christaller's model, the network is a deterministic fractal: the designs do not undergo any deformation. In reality, the regularity of urban networks is deformed according to the relief, the presence of a stream, or even the development of an industrial or tourist town. Deterministic fractality is then replaced by probabilistic fractality. In every case, however, the fractal design of towns is quite real.

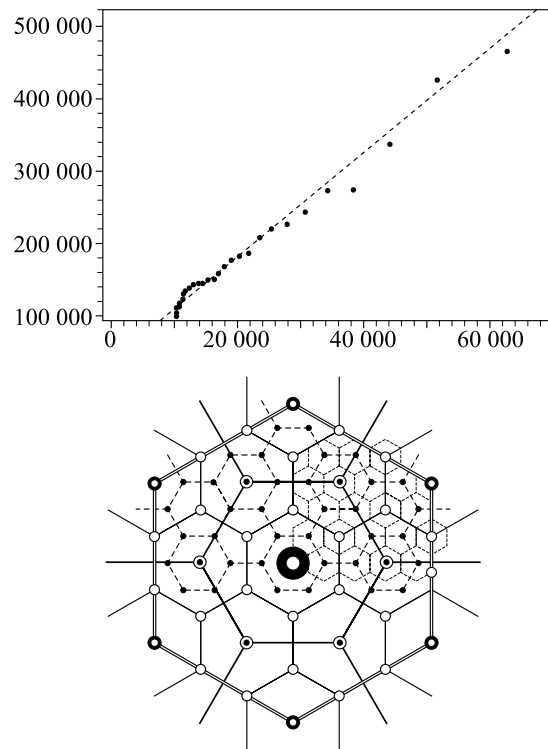
Thus, urban networks properly present a double fractal signature. Contrary to the thinking of a number of geographers, who disagreed with the discontinuity of hexagons and the continuity of the rank-size distribution, both of these distributions are facets of one and the same mechanism – critical self-organization.

This “rediscovery”, which attributes a common mechanism to the Christaller model and the Auerbach-Zipf distribution, is noticeable, but it is still insufficiently understood in disciplinary terms. Fractality and criticality are very closely linked. It is difficult, however, to specify the modalities of this relationship, or integration.

It is possible for criticality to be obtained through a synchronization tendency. An initial local network of hamlets and villages is formed around a market town. Then, once agglomerated to other local networks, through synchronization they historically provided a regional network. The passage from a local to a regional network is synonymous with bifurcation, with overcoming a critical threshold. This

same process is repeated on a national scale, during the formation of nation states, with synchronization of regional urban networks.

We would also be in the process of witnessing this synchronization on a global scale, with the emergence of a world network. This type of reasoning is found in Y. Chen's complete study [CHE 09], which compared urban and river networks. It underlined, as we did, the iteration of these successive synchronizations for which the role was repeated in the creation of fractal forms.



**Figure 8.2.** *The two fractal signatures of urban networks*

This type of emergence through synchronization in critical self-organized systems is often described in physics. A long period of stability is often followed by a rapid reorganization, followed by another phase of stability. The metastable urban system reaches a critical point before being directed towards another metastable state. The hexagonal and relatively regular fractality of regional and national

networks is, however, without doubt similar to fractality, which is still poorly understood and irregular and provides order in a global town network. The unequal distribution of continents and oceans, but also the history of the great civilizations, has destroyed hexagonal fractality.

Thus, the economic theories of Christaller and Loesch explained the spatial hierarchy of town networks; whereas other similar theories, calling upon endogenous, exogenous or mixed development, managed with more difficulty to account for the rank-size distribution. Critical self-organization theory embraces both fractal distributions in a single explanation. It generalizes previous knowledge and reunites what was separate.

Whatever its imperfections, it offers a new vision of a town network. Among its imperfections, we must admit that this theory does not provide any precision beyond double fractality. In order to understand the hexagonal form of networks and the precise localization of towns, other principles are required that are, without doubt, economical, political or physical.

In order to explain critical self-organization, other authors have advanced new hypotheses on dynamic synchronization. Thus, for D. Sornette [SOR 99], human societies are removed from a stable state by a set of positive self-regulations. Pockets of innovation, but also of speculation, form the basis of success leading to further success, but these pockets – which provoke exponential growth – are unable to last. They are therefore followed by crises leading to a new state of equilibrium, before a new pocket emerges.

These pockets have punctuated the history of humanity. The development of the railway in the United Kingdom, the project to sequence the human genome and the dot-com bubble are examples of this accelerated growth, followed by a crash. Some giants survive these crashes, for example Yahoo and Google, who forced a very large number of companies into bankruptcy. These mechanisms challenge the rational behavior of participants, who are lauded by liberal or Marxist economies, even if they bring to light group behaviors.

In mathematical terms, these repetitive human behaviors attach great significance to exceptional events. They cause divergence and lead to power laws. More precisely, this mechanism would account for the frequent log-periodic distributions [YAN 10] during the evolution of a system.

A third interpretation, which has been expressed in various forms, attributes the fractality of a critical self-organized system to the interaction of two opposing processes. Thus, the evolution of a pile of sand, which is often cited as being a critical self-organized system, is produced by the interaction between gravity –

which acts towards the base – and the addition of new grains of sand – which make the pile grow. We have noted that this opposition of two forces was also invoked by H. Le Bras [LEB 00] as an explanation for the fractal distribution of a population.

Above all, these dual interactions account for the on–off intermittence noticed within turbulent flows. This intermittence provokes a succession of stable phases, pervaded with gusts of very intense irregularity. This generalized intermittence mechanism is not without some links to N. Eldredge and P. Gould’s theory of punctuated equilibrium [ELD 88]. This theory better explains the fractality of ecological areas than the Darwinian theories that advanced the principle of regular and gradual evolution.

Thanks to these links to punctuated equilibrium, critical self-organization theory is often used to account for ecological phenomena. L. Seuront and N. Splimont [SEU 00] thus explained the distribution of microphytobenthos in the intertidal zone. This theory is also being used to further the understanding of how the introduction of a new species into a medium may create fluctuations that provoke extinctions in cascades.

The list of applications of critical self-organized systems theory is constantly being enriched. Yet, from a practical viewpoint it always leads to fractal phenomena observed in time and in space.

### **8.3.2. *Béjan’s constructal theory***

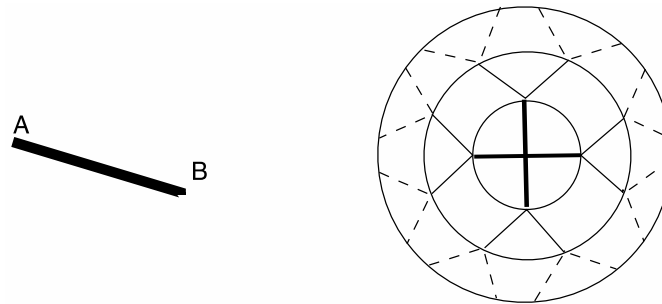
A mechanics specialist, A. Béjan [BEJ 00] also accepts the fractality of individual phenomena and on a global level, but he provides a different explanation. First, the world is built via aggregation from elementary building blocks; hence the term *constructal*, which is used to describe his theory. Also according to Béjan, this hierarchical assembly occurs based on a principle of optimization, which is an example of the least action principle.

This type of logic is not completely new. It previously guided research on the evolution of biological forms conducted by D’Arcy Thompson. For this great biologist, however, the optimization of forms was the result of physical forces, not energetic constraints. This type of logic, which was clarified by the consideration of recent developments in physics, is still favored by V. Fleury [FLE 09] to describe living forms.

For A. Béjan, however, the optimization of forms is energy based. For a system to be self-sufficient, it must evolve in such a way as to enable an easy, optimal circulation of the flows that cross it. Thus, flow systems – such as blood, river or

transport networks – evolve in order to construct the most efficient forms. Transport systems therefore move towards their overall objective, which is to maximize traffic flows.

For example, optimization of the flow between two points requires a single linear tube (see Figure 8.3a). For flows directed from a central point towards a circle, the network is fashioned according to Figure 8.3b, which is a hierarchical and fractal tree. This tree constitutes the optimal form for the separation of a liquid that flows between a point and a surface. This logic remains valid for commuters, who travel from the periphery of town to the center or back the other way. Various physical distributions, in particular the Murray rule, provide the number and types of joints, and even the dimensions of the branches of the network. These values are dependent on the circulation rate of the fluid, people or information.



**Figure 8.3.** *Optimization of networks using constructal theory*

The formation of fractal hierarchical networks is well explained by this theory. The geometry of these networks is not due to chance, but it is determined by simple laws that enable the best result to be attained. This theory is well-known to engineers, who apply it to transport networks. On a global level, it is invoked in order for us to understand the formation of atmospheric circulation cells, the fractal organization of town roads, river networks and the granular structure of beaches [REY 10]. It is also invoked in order to understand the scale invariance laws that link displacement speed to body shape for all animals.

Although it is not well-known in geography, like most of the theories based on a principle of optimization, it should inspire the land planning specialists. On one hand, they are devoting a great deal of effort on the transport and communication axes, but sometimes neglect the network effects. On the other hand, they need to think of land in terms of optimization and equity.

Beyond these practical applications (see Chapter 9), the ambition of this theory is to complete, or even replace, theories explaining morphogenesis through single functions. For example, in a living field, V. Fleury's theory – like that of D'Arcy Thompson – is accorded a secondary place to genetics. Fleury has very strongly insisted that Darwinian schemas have gaps. More generally, these theories do not apply *a priori* to forms in which flows that fulfill functions circulate, but forms appear and evolve in order to optimally distribute flows. Whatever its limitations, this theory has contributed to the regeneration of morphological sciences. It is this which should inspire geographers and developers. Doubtless it does not enable an understanding of all of the irregularities of a network form, but it does take account of its overall structure. Thus, this theory sheds new light on areolar organization, while at the same time being directed by transport axes and urban spread.

### 8.3.3. Nottale's scale relativity theory

L. Nottale's scale relativity theory [NOT 98] is original in more than one respect. First, the principle at the heart of this theory is easy to understand; it is the principle of relativity. Galileo introduced positional relativity and Einstein the relativity of movement. A position or movement only exists relative to another position or another movement. L. Nottale proposed that this principle should be applied not only to movement, but also to scale. Bear in mind that this principle of relativity should not, in any case, be confused with philosophical relativism. In fact, it teaches the opposite.

This principle of relativity certifies that the laws of nature are valid for all reference systems. They apply whatever the type of movement and whatever the state of scale of the reference system. However, as one position only exists relative to another, and one movement relative to another state of movement, we are speaking in terms of relativity. Movement, as with position and scale, is not defined *per se*, hence the term relativity, which does not imply any philosophical relativism.

Second, moving away from this principle, L. Nottale [CHA 09] and his collaborators have reconstructed a new theory of the world. The mathematical formalism of this theory falls outside the scope of this book and also outside the modest competencies of its author. However, it is possible – thanks to reading numerous publications – to retain some useful laws for geographers:

- Firstly, space–time – the curve used in Einstein's theory – is necessarily fractal in scale relativity theory. This link therefore unites scale relativity theory and world fractality. Simply put, instead of defining a variable through its coordinates relative to a single time, i.e.  $x(t)$ , it is written as  $x(t, dt)$ , where  $dt$  is the scale variable of time. Also,  $x(t, dt)$  may become a fractal coordinate. In concrete terms, the usual

physical size – for example a population within a demographical study or amount of rainfall in climatology – is replaced by a function that explicitly depends on observational resolution.

– Second, as the state of a movement is described by a speed, the state of scale is defined by spatiotemporal resolutions, i.e. a fractal speed field. It is then possible to deduce a concept similar to acceleration from this. This information will appear quite theoretical for most geographers, even more so when the mathematical formalism of this theory remains difficult to access.

There are other more pragmatic conclusions to be contemplated. Two of them principally call the works of geographers into question. On one hand, the authors and practitioners of this theory simulate very realistic morphogenesis using a generalized Schrodinger equation that formalizes a movement. Unfortunately, the only available example remains the blooming of a flower. On the other hand, L. Nottale has shown that the log-periodic distribution is deduced from scale relativity theory. This therefore contradicts W. Newman [NEW 95], for whom a log-periodic distribution is simply the realization of a critical self-organization process over a fractal space. In this way, L. Nottale explained that the fractality of fault fields is the basis of numerous log-periodic distributions discovered by studies into earthquakes. Hence in Chapter 9 we will present a number of predictive and decisional studies that make use of this log-periodic distribution, which is easy to test using experimental data.

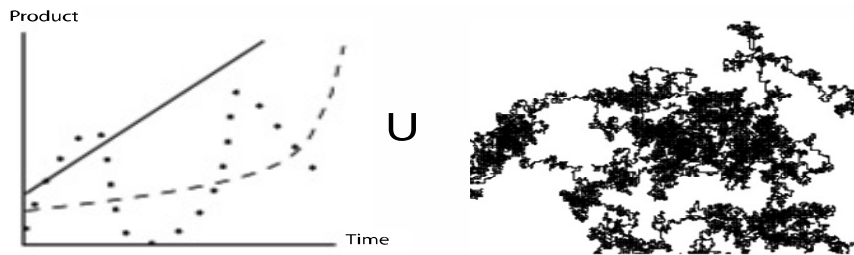
With the exceptions of M. Forriez's thesis [FOR 10] and the works of P. Martin, few French geographers have bought into this theory, which appears to work well for spatial analysis problems. In spite of everything, they should – as with all theory – define its limitations, i.e. what has not yet been realized. It is all the more necessary therefore for scale relativity theory to remain the subject of lively debate, even within astrophysics, its initial field of application.

#### **8.3.4. *A general theory of morphogenesis***

Here we present a general theory of forms, which is therefore explanatory for fractal forms. It generalizes so-called reaction–diffusion theory. We recall that the Earth, as is the Universe, is an amalgam of forms. These terrestrial forms emerge, are developed at varying speeds and are then wiped away and replaced with other forms. Surfaces of erosion are replacing mountains and towns are covering agricultural areas.

#### 8.3.4.1. *Two factors in an infinity of forms: production and movement*

All geographical forms are the culmination of a double action of dispersion, or energy concentration, of matter, people or ideas. These two contradictory tendencies are the result of a coupling between production and movement. Figure 8.4 illustrates this assembly of two mechanisms. The Union sign (U) does not have any particular significance.



**Figure 8.4.** *Growth associated with random movement*

In this theory, the term “production” describes all types of production: the production of goods, which is beloved of economists; but also the production of a sedimentary layer, a new species, a new illness, new technologies, ideologies, or works of art.

This idea of production is present in all of the sciences, but under different names. Chemists use the term reaction, while ecologists talk about the growth of a population. Beyond this variety of terminology, however, all production is modeled using a growth curve. This growth may be positive or negative. Certainly production mechanisms are very different, but they are all represented by a growth curve that is formalized by a differential equation, which is expressed in a very simple form:

$$\partial p / \partial t = r.p \quad [8.1]$$

where  $p$  may be the volumes of sediment, number of cars, people or geographic articles published in a magazine, and  $r$  is the growth rate.

This growth equation presents two fundamental features:

- It is not integrated in space. This assessment has a profound significance for geographers: no form could arise from growth alone, which is why self-organization

theory, which is modeled with its own equations, does not manage to simulate morphogenesis.

– The creation of a terrestrial form, by necessity, requires a second ingredient – movement – which is explained below.

Let us emphasize that previously there have been not one, but a great number of growth distributions, as the phrase “growth rate” covers numerous mechanisms, particularly types of linear, exponential, logistical and cyclical growth. Figure 8.4 only represents simple examples of this. R. Banks’ book [BAN 94] describes these types of growth in more detail. Only logistical growth, if present in ecology or in economic studies devoted to innovations in material and cultural assets, models numerous behaviors. Overall, as far as geographers are concerned, growth is an initial source of infinite diversity.

So that forms can emerge and be structured according to geographic territory, all product of growth must be redistributed into space, which implies movement. Movement is observed as much in the physical world – in air flows that are expelled from an anticyclone, water or energy flows from a river – as in animal and human society. If blocks of rock roll down slopes, the whole town is animated by the incessant displacement of its inhabitants and by information flow. In fact, everything is movement. Heraclites had already pointed this out.

Yet, all elementary movement, such as diffusive movement, is formalized by a partial differential equation expressed as:

$$\partial p / \partial t = -D \cdot \partial^2 p / \partial x^2 \quad [8.2]$$

where  $D$  is a diffusion coefficient and  $x^2$  represents the two dimensions of a flat space.

Contrary to the growth equation, this equation takes space directly into account. The first equation of this type was introduced by Fourier in order to explain the movement of heat, from warm to cold. There is no movement without spatial dimension; all movement has direction, which may be random, for example in the example of Brownian motion. However, it is often oriented. In the Northern Hemisphere, tradewinds from the tropical zone transport masses of air from East to West. As far as pendular movements are concerned, they simply displace employees from the outskirts to the town centre in the morning, and then back again in the evening. These random or oriented movements form the basis of terrestrial forms. For example, flood waters hollow out the flood plains and Christianity has covered European territories with churches and chapels.

By retaining distance, as the only discriminating criteria, over and above the numerous differences due to their nature, the world is covered using two main types of movement: diffusion and advection.

*Diffusion* corresponds to a movement over very short distances, which barely exceeds the sizes of the elements considered [SAI 85]. In physics and chemistry, diffusion is above all applied to the movement of molecules. This has often been formalized, using the Fick distribution, before Brownian motion invested time in microsimulation models.

In human geography, it is possible to talk about building diffusion in a town, or the diffusion of a disease within a population [BAT 94, SAN 01]. When it acts alone, diffusion irons out spatial heterogeneities. Therefore, diffusion does not create, but tends to remove forms, particularly gradient forms. It balances out the pressure between anticyclones and depressions, which is why movement itself is not enough to create a terrestrial form. Rather, a form is the combination of growth and movement.

Conversely, the terms *advection*, for horizontal movements, or *convection*, for vertical movements, are used to describe movements made over a long range compared with the sizes of the entities considered. Thus, international migrations are advectations; while daily, intraurban displacements are similar to a diffusion mechanism.

This advection occurs in the form of laminar or turbulent flows. However, laminar flows, which are widely studied in physics, are exceptional. These are observed over “average” distances and reduced time intervals. Indeed, along the scale considered, flows are always affected by turbulence. If the Gulf Stream appears as a laminar flow across the Atlantic basin, satellite photographs reveal strong turbulence on a wider scale.

These movements are rarely free, but a flow occurs over a porous material. In this case, movement from the restricted diffusion through a substrate is described as percolation. Percolation has resulted in a vast amount of literature [LAG 03]. A percolation model is therefore a combination of the displacement of one variable relative to another, which is mostly considered fixed, although this is not mandatory.

The diffusion of uncultivated land over agricultural terrain or the development of urban habitats in a rural space is similar to a percolation mechanism. The diffusion of Hispanic families within English-speaking communities in North American towns is another example of percolation. This type of logic is also guiding the introduction of a new artistic sensibility in a world dominated by other forms of representation. This physical model is therefore transferable to cultural geography, to explain, for

example, the appeal of the impressionists to a clientele steeped in classical or romantic values.

In 1957, a mathematical model formalized this percolating flow. It demonstrated that below a threshold,  $p_c$ , known as the percolation threshold, pores that allow liquids to pass through are clustered in a basic form. Also, the sizes of these clusters increase when the percentage of pores,  $p$ , approaches  $p_c$ . Above this threshold, an infinite cluster of pores, which is continuous over the considered surface, is set up and increases as and when the percentage of pores,  $p$ , increases. Thus, below the value  $p_c$ , the liquid is blocked. At and above the threshold, however, it manages to find a way to pass through the porous material. Percolation is therefore a connection model between the different elements of a set, which are interlinked partially and randomly.

So percolation is a source of fractality. For solid structure pores, it has been demonstrated that the percolation cluster at the threshold  $p_c$  possesses a mass fractal dimension slightly below 1.89. The outline of this cluster is also a fractal curve, with a dimension equal to 1.62 within a two-dimensional space. Finally, this fractality characterizes the cluster perimeter and the percolation front.

As with diffusion models, many physicists have developed models that have been derived by introducing supplementary constraints found in daily life. A first example refers to the process of invasion percolation, where two fluids are brought together and a front is displaced between them. The first fluid progresses due to a force and presses the second fluid, enclosing it in incompressible areas from which it cannot be dislodged. This type of model has practical applications, such as oil pumping after extraction from a deposit. It is, in fact, possible to improve the extraction of the crude oil by injecting a second fluid under pressure into the ground water. The dynamics of military territorial conquests appear to be catching up with this. More generally, when the substrate evolves over time, this should be referred to as dynamic percolation. In addition to this, in the initial theory there is no interaction between a pore and its neighbors, which is common in reality. New models are therefore simulated using multiagent systems.

In summary, the morphogenesis of the whole spatial entity, whether from the formation of a sedimentary layer on the ocean floor or the spread of a large town, will be defined by the following general equation:

$$\text{Variation in } p = \text{growth of } p + \text{movement of } p$$

All geography, assimilated to a morphogenesis, is only, at the end of the day, a reflection on this fundamental equation.

These two principles form the basis of an infinite variety of theoretical forms, some of which are observed on the Earth's surface. For example, gradient forms – which are so common in climatology – are associated with the production of high and low pressure and diffusion and advection–convection movements. These same ingredients are observed in the polarized forms, which are common in human geography.

Growth and movement are also the basis for urban forms, particularly segregation areas. Spiraled and networked forms are combinations of growth and movement. This is quite clearly the case for all fronts observed in climatology or oceanography, but also for urban or pioneering fronts. When a form emerges, it in turn guides and constrains morphogenesis, either in theoretical models, such as the Schelling model, or in reality.

Finally, the production–diffusion pairing would explain the rank-size distribution of urban hierarchies. D. H. Zanette and S. C. Manrubia [ZAN 97] simulated the setting up of an urban hierarchy using just these two ingredients. They compared the results obtained using rank-size distributions for towns in Switzerland, the United States and worldwide. They showed that it is sufficient to use these two systems, although they did attach a preponderant weighting to the production–reaction effect.

#### 8.3.4.2. *Physico-energetic constraints, natural selection and optimization reducing the infinite number of potential forms*

There are fewer forms observed than those generated from the innumerable simulations that can possibly be produced with this elementary equation and its generalizations. A mechanism therefore reduces this infinite profusion. It carries three different names, depending on whether inanimate, living or social forms are being handled.

The first, for example a relief form, is subjected to energetic or physical constraints, such as gravity. A second mechanism, Darwin's natural selection, limits the number of living forms. Finally, the principle of optimization guides human activity. It further reduces the range of possibilities and eliminates forms.

All of these principles together represent, in a way at least, the action principle. Nowadays physicists prefer the more neutral phrase variational principle. Ultimately, no humanized space can escape gravity, natural selection and a certain type of economic and sociocultural optimization, which varies according to the dominant values of each civilization.

#### 8.3.4.3. *Some initial preponderant conditions*

This growth and movement process is not carried out on a homogeneous or random starting area, but on an area that has already been structured. All quaternary forms of relief are derived from previous forms and contemporary towns are developed from ancient, then medieval towns and their surrounding countryside. Terrestrial space is a structure, a memory that constrains the two mechanisms of growth and movement, while being the result of the same two mechanisms.

A great many flows only use pre-existing networks. River water and car traffic flows are thus trapped within these networks which they, in return, help to shape. It is therefore possible to borrow the Fitzhugh-Nagumo model, which was established in order to understand the propagation of the nervous system in the axons. Simulating equations that are associated with growth and movement in a network produce this dynamic quite well.

#### 8.3.4.4. *A theory preceded by innumerable models*

Despite the originality of this theoretical essay, we are not the first scientists to have grasped the significance of the coupling between production, represented by growth and movement. For a century, numerous models have been put forward in different fields of knowledge. These have explained the dynamics of microforms encountered by geneticists or macroforms that are observed by astrophysicists. Most of these models are associated with a growth and movement mechanism. The terms used vary according to the discipline, but they still describe these two mechanisms.

Thus, the so-called reaction–diffusion model, which has guided our theoretical reflection, was discovered by many researchers who considered scientific fields that were very distant. According to J.D. Murray [MUR 03], the first equation was proposed by Luther in 1906. Then, in the 1930s, the ecologist J. Fisher and the physicist A. Kolmogorov applied this theoretical model to genetics. They analyzed the localization, type and propagation of fronts emerging based on the rate of diffusion. This theory owes its name to the information mathematician A. Turing [TUR 52], who sought to understand the formation of stationary structures in real life.

Each researcher showed how a two-equation system, coupling logistic growth with normal diffusion – which geographers call contagion – is sensitive to miniscule fluctuations. Later, the biologist H. Meinhardt [MEI 82] applied this same model to life sciences, using the term activator–inhibitor to explain the morphogenesis of numerous species, particularly spiral or shell-shaped forms. These models integrated different growth equations, which became increasingly “realistic”. In effect, he turned down this theoretical model by “manipulating” increasingly complicated

growth equations, but he always associated them with a simple normal distribution modeled using the Fick distribution.

Another way of modelization that is adopted by physicists, notably Ben-Avraham [BEN 00], consists of complexifying movement, particularly diffusion, rather than growth. Normal diffusion is equivalent to a Brownian model. Now, it is easy to simulate abnormal diffusions, for instance those carried in a convection current, some of which resemble the hierarchical diffusion from geography. We have already shown this in previous chapters, with the Lévy and fractional Brownian models.

Lévy flights, where multiple displacements over very long distances deform the Brownian model, are often used. The international relocation of businesses is an example of this. More recently, some models have represented displacements through matching a diffusion process with an advection (Burger equation).

In spite of the number of studies on turbulence produced by physicists, there are few examples of research matching turbulent movements with growth. However, this type of approach is not impossible, since many studies carried out without dimensions by Damköhler and Péclet already enable a comparison of times between reaction or diffusion and advection.

Furthermore, different reaction–diffusion–turbulence models have been put together to account for the multifractal spatial organization of marine plankton. More generally, ecologists have been integrating increasingly “realistic” constraints in order to simulate movement of species and there are countless studies on this subject [OKU 01]. They take into account the intentionality of displacements, their irregularity between individuals, or the conflicting effects observed in flocks of birds or shoals of fish. Modelization using cellular automata or multiagent systems is therefore favored over modelization using differential equations.

In the social sciences, this theory is also well demonstrated, but the link to previous works in other disciplines is rarely established. For example Schelling’s theory [SCH 80], which aims to explain sociospatial segregation, combines these two mechanisms. Diffusion, which is normal in the baseline model, is induced by cultural choices. Young researchers have been introducing more complex diffusion mechanisms formalized using multiagent systems [BAD 09].

We, ourselves, presented H. Meinhardt’s activator–inhibitor model to the *Journées Européennes de Géographie Théorique et Quantitative* (European Symposium on Theoretical and Quantitative Geography) in 1983, without totally perceiving its quality. Later, we proposed this same theory in order to explain the “leopard skin” development observed in Italy [DAU99]. Above all, E. H. Decker

[DEC 07] managed to simulate a rank-size distribution from two human population reproduction and migration mechanisms.

#### 8.3.4.5. *A theory on fractality and the “adjustment of levels”*

The normal diffusion equation is invariant through scale transformation [LAG 03]. Forms of normal diffusion and their fronts are auto-similar, while the temporal change in the number of particles that are diffused is self-affine. Thus, total diffusion implies fractality of the resulting forms.

This result remains valid for forms generated through turbulence. This forms the basis of multifractal forms observed in all turbulence flows. Navier-Stokes equations, which are nonlinear partial derivative equations, are also invariant through scale transformation. More precisely, this is intermittence in an ordinary dynamic, and therefore an interaction between phenomena at different levels, which generates multifractality.

This is why the fractality of natural forms, and of the majority of forms created by societies, is not the exception but the rule. Indeed, diffusion and turbulence nearly always characterize movement. Quite clearly, this fractality is also produced by more complex movements (Burger and Klein-Gordon equations). These initial remarks explain the generalization of fractal forms and therefore the permanent adjustment of levels in geography devoted to morphogenesis.

This adjustment only causes growth if we accept that turbulence is more frequently present in modern-day societies – where mobility is accelerated – than in ancient civilizations. This is still to be demonstrated, but it constitutes a plausible hypothesis. It is certainly possible that new forms of development, which are less energy-hungry, are reducing Man’s mobility, but the mobility of information is rapidly increasing and this trend shows no signs of abating in the near future.

Thus, fractalization and globalization are both shaping our future world, but globalization is not synonymous with homogenization. On the contrary, it is provoking an increase, generated from fractality, in all flows. As a result of globalization, the world is fragmenting, and our theory takes perfect account of this paradox.

Admittedly, like all theories, this one remains incomplete. It attaches little importance to the perception of these forms by participants. Furthermore, it partially answers the question of *where*, which is always relative, even if it is possible to simulate the localization of certain emerging forms, particularly contact forms, such as discontinuities and fronts.

Indeed, repeated simulations show that the emergence and precise localization of poles are based on the difference between diffusion coefficient values. These emerging localizations have been widely studied in biological activator–inhibitor models. In spite of these imperfections, it does not remain any less true that the formation of geographical spaces, and their dynamics, are shaped by this metalogic.

### **8.3.5. *Chaos and fractal analysis theory***

For the past 30 years, chaos systems have been observed across all disciplines. Geography, and in particular physical geography, has not escaped this trend [DAU 95]. Given the choice between random behavior and deterministic systems, chaos systems are deterministic and can therefore be formalized by differential mathematical equations. However, their behavior mimics a random evolution and rapidly becomes unpredictable. This rapidity is dependent on the system being considered. This is counted in billions of years for the solar system, but in days for changes in the weather.

This chaotic evolution, which has been widely studied across all disciplines, characterizes systems that are sensitive to initial conditions. The game of billiards is a typical example. From virtually identical initial trajectories, after several displacements a billiard ball can be directed towards totally different areas. Everything is in the “almost” and new chaotic behaviors are constantly being discovered.

Among those of direct interest to geographers, we mention all of the turbulent systems that characterize changes in the weather, such as oceanic currents and mantle convections, which therefore influence tectonics and relief forms. As well as this, we can include crowd movements, stampedes and many other types of displacements, particularly traffic flows during great vacation migrations.

The passage from predictable deterministic to chaotic behavior, which is therefore deterministic but unpredictable, happens according to different modalities, the analysis of which falls outside the scope of this book [DAU 03a]. This chaotic evolution is continually the result of successive bifurcations.

It is, however, possible to distinguish between systems for which behavior is uniquely chaotic in time and those that are chaotic in both time and space. The latter corresponds to generalized turbulence in physical systems, such as human systems. Both of these categories of chaos systems have something in common; their phase space is a fractal.

In order to demonstrate this, based on the data from a chronicle series representing the evolution of the system, geographers have put together the equivalent of a phase space. They report on a graph for which the coordinates plotted are  $X_t$  and  $X_{t+n}$ . The value of  $n$  is determined empirically, taking the values 1, 2, 3, etc., or by noting the abrupt decline in the calculated auto-correlation over the series of data handled.

Disordered phase space is synonymous with random behavior. Conversely, when the phase space is a fractal (see Figure 1.3), the behavior of the system is chaotic. Thus, the fractal analysis of the constructed phase space from temporal data is used to simultaneously test the calculation of Lyapounov coefficients or various entropies and to differentiate between chaotic and random behavior. For example, in astrophysics the Hénon attractor has a Hausdorff fractal dimension equal to 1.26.

Additionally, in the case of turbulent, and therefore chaotic, systems in space and time, the forms produced are themselves fractals. The fractality of these turbulent systems is observed not only over phase spaces, but also in forms that emerge and are then subjected to highly complex alterations, which are not random but are deterministic. In short, it does not take many variables to understand these systems, but their spatiotemporal behaviors take many forms.

These chaotic behaviors, which generate fractal forms, have become well-known in the natural sciences, in oceanography as well as in meteorology. Without doubt they are also all commonplace in society, but they are less frequently studied and less often noticed. Despite everything, once again fractality and complexity appear as two facets of the same reality.

#### 8.4. Conclusion

Complexity theories are providing new, more general and independent explanations of precise disciplinary contexts. Thus, the four theories presented in this chapter explain all of the power laws encountered by geographers, and also those encountered in linguistics, geophysics or in the life sciences, which is why these abstracts are supradisciplinary formal metatheories.

Geographers tend to be embarrassed, or even frankly hostile, about these highly abstract constructions. They are always putting forward the same arguments: the innumerable interactions by agents and man's freedom of movement. Whatever the civilization, the ideology of leaders and the ensemble of participants, however, seaside cities are nearly always built using the same model. This type of reasoning is also applied to numerous objects. Thus, P. Grou [CHA 09], rightly remarked that in a quasi-identical way, four civilizations were founded in the same era in Egypt,

Mesopotamia, China and India, based around a great river. The geographical freedom of man is expressed within this physical, biological and societal metalogic.

Moreover, all of these theories cause forms to grow and it is difficult to favor one over the other. Some supplementary research turns out to be necessary; however movement is the common ingredient in these four theories. Yet, all movement equations are scale invariant; they generate fractality. To live in a moving world is to live in a fractal one.



## Chapter 9

# Land-use Planning and Managing a Fractal Environment

“In a fractal market, too many big variations spring up when you’re not expecting them, whereas, the rest of the time, we spend time waiting for average variations which never happen.”

Jacques Lévy Véhel and Christian Walter, 2002

While the fractal approach enables a better understanding of territories, it is no less useful in applied geography, land-use planning and environmental management. As with previous chapters, we make no claims of being exhaustive in our treatment of applications. Our more modest objective is to lay down some markers in exploring some examples that specialists might seek to develop further.

Fractals present three essential properties:

- First, they draw the geographer’s attention to extreme values. In this way they become an essential tool in risk management.
- Second, being able to locate irregularities in time or space makes fractals a useful tool for segmenting and identifying forms in image processing. Object boundaries are effectively major irregularities. In this respect, their scope has

recently been widened. Various applications have become routine in medical imaging, but they are still unusual in remote sensing.

– Third, fractals often respond to optimization mechanisms. They enable the largest surface inside a volume or the longest line within a plane to be noted. They are therefore useful in all environmental or town issues calling for an optimal solution.

This chapter is arranged around these three axes, before tackling potential landscape analysis applications, since fractality is without doubt linked to the human perception of beauty, as is indicated by the surge in popularity of fractal art.

## **9.1. Fractals, extreme values and risk**

There are many studies on the application of the power laws to extreme events in a series. Few of these have a directly operational objective, but they are an indirectly rich source of information in reducing the dangers that exist in modern-day society. This information leads to fairly pessimistic conclusions.

### **9.1.1. *Under-estimated hazards in preliminary risk assessments***

The first piece of information is that power laws are synonymous with persistence in time and autocorrelation in space. Indeed, a power law presents a wide or thick tail [DOU 03, MEY 08]. The decrease in the power law is slower than that of an exponential distribution, which is commonly used to estimate the intensity of events corresponding to a given return period, whether this refers to a decadal flood or an earthquake.

This rule signifies that extreme events occur more frequently than we are led to believe from other adjustments, particularly the Gumbel exponential adjustment and the log-Pearson 3 exponential adjustment, which was used by the Water Resources Council in the United States, or the generalized logistic distribution favored by service providers in the United Kingdom [MAL 06].

We are therefore at a greater risk of experiencing hydrological floods than envisaged by classical studies. The seeds of this idea had already been planted in the minds of the authorities during the building of the Serre-Ponçon dam and after the Nîmes disaster, following a multifractal analysis of rainfall at the Nîmes-Courbessac climatological station [LAD 93]. This rule is now accepted by the French and European hydrological services concerned, notably for basins that are poorly equipped with measurement instruments. They have rejected Gumbel's distribution in favor of Fréchet's, for which the probability distribution decreases according to a

power law rather than according to an exponential distribution, as is the case for a Gumbel distribution. Beyond these technical refinements, geographers maintain that, for a given intensity threshold, the risk return period is much shorter. Dangerous hazards therefore clearly occur much more frequently than estimated by Gumbel distribution.

This type of logic is applicable to other natural hazards, such as the earthquakes observed in the United States between 1769 and 1989. By proceeding to an adjustment using a Gumbel distribution of the most violent earthquakes, there is only a 2% probability of an earthquake of magnitude greater than 8 being recorded. However, the 10% threshold is passed for a Pareto distribution, i.e. a probability that is five times greater than given by Gumbel distribution. Similar results are characteristic of wind speeds, which can exceed 30 km/h in Paris.

Similarly, in financial management, fractal financial risk modelization is beginning to gain followers. This choice has been the topic of some very lively debates since the 1980s. Financial techniques have made use of increasingly complex Gaussian modeling, before some financiers used  $\alpha$ -stable distributions. The lower the value of  $\alpha$ , approaching 0; the stronger the probability of large jumps, and therefore gains or high losses, occurring. The scaling parameter  $\alpha$  is therefore a good risk indicator for the series under consideration, whether this refers to the course of an activity or some other indicator. It measures the capacity of any chronicle series to present a discontinuous trajectory. Financiers say that the volatility of the product increases. Practical advice on management follows from this, notably on the choice of optimal portfolios based on the degree of risk incurred.

Indeed, beyond financial analysis, risk adjustment using an  $\alpha$ -stable distribution enables form risk to be distinguished from risk due to intensity [LEV 02]. The former, which describes the fairly high capacity of the process studied to go wrong, is measured by the scaling parameter  $\alpha$ . The latter, which measures the intensity of these distortions, is quantified by the scale parameter,  $\gamma$ .

This very general type of thinking should be applied to all natural or technical risk, or even biological risk with the mutation of viruses and bacteria. We do not know of any studies based on this approach, however, apart from in the geosciences and in finance; it does appear, though, that the major insurance companies are interested.

Sometimes fractal analysis leads to optimistic conclusions. In technological risk, when the lifespan of an instrument or a simple part behaves like a Pareto variable, that signifies that the greater use it has undergone, the more chance it has of an extended life. Power laws, when applied to a lifespan, are in a way synonymous

with rejuvenation; accidents thus become more unlikely, at least in the temporal context considered, since these accidents end up occurring.

This paradoxical rejuvenation law is also observed in demographic curves, with a reduction in increase in the mortality rate from a certain age. The aging of the French population will again be more noticeable under these conditions than when left to be assumed using the classical approach built around mortality tables. Important economic, social and political consequences follow from this.

Multifractal techniques also provide a new interpretation of the limits of predictability of nonlinear processes. Chaos theory announces a rapid, but rather regular, loss of predictability, and yet the multifractal analysis of complex spatiotemporal systems highlights that there is no characteristic time for loss of predictability; this follows the power laws, not exponential distributions. The role of intermittence is then very effective and the loss of information, and therefore of predictability, occurs in bursts and not regularly. In concrete terms – in meteorology for instance – this rule indicates that predictability is not only unequal – depending on the type of weather – but it can be highly irregular, or even turbulent. No doubt, the same applies to a number of risks, for example during the explosion of cases of illness during epidemics.

Thus, predicting the weather, which according to chaos theory becomes increasingly unreliable beyond seven days, can be improved beyond this threshold, or can, conversely, become more precise much more rapidly according to the amount of atmospheric intermittence. This type of reasoning remains valid for other turbulent phenomena, such as traffic during the great summer migrations. Regulation and management of these events are therefore very complicated, with pleasantly clear roads suddenly turning into endless traffic jams.

More recently, great store has been placed in the use of log-periodic distribution. D. Sornette [SOR 98] applied this distribution to numerous seismic and financial crises. He first showed that similar earthquakes followed this distribution. Some chemical anomalies, with a chlorine ion obeying this log-periodic evolution, were analyzed following the Kobe earthquake.

Some attempts at using these distributions have been carried out based on events preceding an earthquake or financial crisis. Some perfectly reliable predictions have not yet come to pass, but encouraging progress has been observed. With a model of this type, various researchers anticipated the observed peak in the increase in oil prices in July 2008 and the bursting of the financial bubble in Shanghai in August 2009. This bubble model, which formed and then burst, was very general and has guided the evolution of our society.

Another solution for improving predictions in the field of risk that has been tested in financial analysis consists of creating a library of fractal forms from very short duration evolutions. Thanks to developments in information, it is easy to compare an instant evolution with this forms library in order to spot the similarities and then to draw conclusions. This type of approach should enable the very short term improvement in predictions, or even the anticipation of disasters.

These few examples show that the distribution of the relevant phenomenon must be determined very precisely. A simple fractal, derived from a linear adjustment, signifies that big events occur frequently. They are much less common for a parabolic fractal, whose curve is negative.

In concrete terms, where the spread of miners' cottages is arranged according to a linear fractal, the future is assured with some significant reservations. It is bleaker for a parabolic fractal with a negative curve. Indeed, the biggest deposit measured over a parabolic fractal is clearly not as significant as that indicated by a linear fractal. S.-Y. Xie *et al.* [XIE 09] successfully applied this technique to estimating the wealth of mining deposits in the Qulong region of Tibet. This same reasoning, deduced from the rank-size distribution, was applied to copper and zinc reserves in India. Other studies [LAH 00] confirm that the oil deposits discovered are ordered according to a parabolic fractal. However, this type of reasoning is portable to other phenomena, in particular most natural hazards.

### **9.1.2. *Fractal networks, fighting epidemics and Internet breakdowns***

The fractal approach opens the door to new strategies for managing epidemics. In order to fight all epidemics, a good understanding of the logic of infection is useful. In the classical models, which were established in the 1930s, the development models for fractal networks – inspired by graph theory – enhance our understanding of mechanisms of propagation. Numerous epidemics, particularly those that promote sexually transmitted diseases, such as AIDS, obey power laws. Yet, this discovery of the fractal distribution influences the choice of matched differential control strategy, which R. Pastor-Satorras [PAS 04] calls targeted immunization.

Faced with an epidemic, it is certainly possible to propose a generalized control plan that addresses everyone susceptible to being infected. For example, this was achieved through the use of a generalized vaccine. The result is effective, as proven by the eradication of smallpox, but is excessively costly. The episode with flu-type A amply demonstrates this. This type of strategy is only acceptable for epidemics leading to certain death for the majority of infected patients.

A second strategy, random vaccination, which is less costly, is nevertheless unsuitable and ineffective as it may be possible to immunize patients who are less susceptible to being infected, while at the same time ignoring people exhibiting high-risk behavior. It was proposed, but happily not applied, in order to curb the spread of AIDS.

The third strategy, which was inspired by the treatment of Internet breakdowns, consists of distinguishing hubs, such as people who often change partners being more likely to contract sexually transmitted diseases, and more generally, people at risk, in order to treat them as a priority – whatever the form of preventative treatment may be. Research of this type is currently taking place in Sweden. Evidently this type of solution, which is optimal in effectiveness terms, will not be perceived as being very ethical by people in certain social categories.

Similar thinking is valid for the internet or the financial markets, which are developed and structured according to fractal logic. Many studies that aim to reduce internet breakdowns have their roots in epidemiological research. The internet does not fear random attacks, as these are not destabilizing. However, the internet is liable to crash in the face of an attack targeted towards a number of hubs. The example of the gigantic collapse of the Iranian network, which was without doubt one of the first publicized acts of cyber war, amply demonstrates this. Furthermore, these breakdowns lead to the collapse of the whole system without warning; since a fractal universe is nonlinear, small causes may provoke large effects.

Different specialists were therefore soon demanding the implementation of a bespoke defense for the internet. “Immunization” must begin with the most frequently visited areas, which are therefore the most likely to transmit a virus. In this domain, in order to counteract the terrorist threat against communications networks, particular attention is paid to hubs. Unlike the treatment of epidemics, there are no ethical concerns about the implementation of such a strategy. Microsoft has recently dealt a fatal blow to one of the biggest distributors of spam emails by applying this strategy in order to destroy the hubs of the Rustock subnetwork.

By transferring this knowledge to the financial world, an immediate conclusion comes to the fore. Beyond ethical considerations, it is not very sensible to allow a global financial network to develop according to a fractal model without putting in some rules principally aimed at the hubs – the great financial centers. Unlike attacks against peripheral economies, such as Greece or Iceland, the collapse of a hub would lead to the collapse of the whole system. Monitoring these centers is a priority in order to avoid a systemic crisis.

As far as businesses are concerned, P. Herlin [HER 10] distinguished between various non-scalable activities, such as the production and sale of pizzas. In this

case, the biggest outputs require the most relevant inputs. The risk level is low and the income obtained regular; this is an example of Gaussian activity. Other activities, however, are scalable. A single product can generate wealth, which is notably the case with pharmaceutical companies or art galleries. Similarly, in the oil industry numerous drillings may produce nothing, but a few discoveries compensate for these efforts. A combination of these two categories of enterprise would be necessary in order to obtain a real and lasting development. According to this author, it would be advisable to defend enterprise capitalism against financial capitalism. While these affirmations, which are based on analogical reasoning, are questionable, there is little doubt that two types of hazard coexist: benign Gaussian and fractal or savage hazards.

In order to become very effective, these analyses imply that network structures must incorporate a temporal delay effect. Epidemics or forest fires do not simply instantaneously explode, but declare themselves following quite a long latency delay. X.-S. Yang [YAN 02] used digital simulations in order to help demonstrate that a temporal delay within a “small world” network considerably modifies the fractal dimension and the probability of extreme events within these networks. The analysis of concrete examples was welcomed.

## **9.2. Fractals, segmentation and identification of objects in image processing**

Fractal and multifractal methods have an immensely wide application to image processing. They are not as frequently used in remote sensing, and still more rarely applied to GIS (Geographical Information Systems), apart from the techniques deduced from geostatistics, particularly the variogram, since the GISs have processing modules matched to these geostatistical tools.

### **9.2.1. *New image processing tools***

In remote sensing and, more generally, image processing, fractal and multifractal methods have an immense field of application. They respond to four common challenges.

#### **9.2.1.1. *Segmenting images in order to identify objects***

First, they enable the localization of irregularities and discontinuities, and therefore the segmentation and emergence of the many objects that comprise them. In this case, multifractal methods are competing with the more classical filtering techniques and some tools from mathematical morphology.

The methods deduced from wavelet decomposition turn out to be the most efficient, as these techniques measure the propagation or reduction of discontinuity singularities and indices at different levels. Thus, for any image it is easy to spot the limits and therefore to proceed to segmentation that enables objects figuring in this image to be identified, or even characterized.

Sciences with the objective of identifying objects containing an image have therefore made use of the fractal approach, coupled with wavelet decomposition. The most frequent works have been produced in medical imaging and astronomy. In medicine, R. Lopes' summary article [LOP 10] describes numerous applications. For example, the fractal approach distinguishes electrocardiograms and encephalograms that contain anomalies, synonyms and illness. Mammograms and cerebral and bone marrow images lend themselves to the same procedures, notably in order to detect microcalcifications and then to better differentiate them from malignant tumors.

In astronomy, multifractal methods are used in order to detect new celestial bodies. They define their boundaries, particularly for double stars – which are difficult to separate – or even faraway galaxies. As these galaxies are often spiral forms, astrophysicists have built new tools that generalize wavelet decomposition in order to better analyze these anisotropic forms. These tools show that ridgelet transforms are optimal for identifying lines, while curvelet transforms are good for isolating filaments. More recently, there have been great hopes expressed for bandlet decomposition and quaternionic wavelet decomposition [SOU 10], but examination of these is outside the scope of this book.

Figure 4.10 illustrates a wavelet decomposition on a satellite photograph of Barcelona. There are still too few examples of this type of processing in geography. An exception to this is the work of A. Lefevbre *et al.* [LEF 09], who spotted changes in agricultural land-use from variations in wavelet coefficients. These researchers have thus brought to light various categories of change.

#### 9.2.1.2. *Differentiating and identifying image textures*

Second, the multifractal approach favors research into and highlighting of textures [MYI 05]. The identification of textures is a still problem that has not been properly resolved. In the analyses that follow this objective, multifractal indicators are compared with Haralick indicators, which are deduced from co-occurrence matrices, spectral techniques and geostatistical tools, particularly auto-correlation. Most of these classical methods ignore the dependency of textures relative to resolution. In order to obtain satisfactory results, it therefore is sometimes necessary to re-perform the same calculations, for example by defining the co-occurrence

matrices for each resolution considered. Wavelet decompositions upstream from multifractal techniques render this replication of the same algorithm useless.

The so-called information dimension,  $Di$ , and the curvature parameter from the multifractal singularity spectrum are considered to be the most efficient indicators. For example,  $Di$  is a measure of heterogeneity. In ecology, a plant formation whose value for  $Di$  is close to 0 will be more heterogeneous than a formation with a higher information dimension value. In the engineering sciences, other studies of this type have led to comparisons of the heterogeneity of materials [SAU 02].

The majority of comparative studies emphasize the low discriminant power of the fractal dimension relative to the lacunarity, calculated on grayscale satellite photographs. This indicator leads to an error rate of less than 15% in the identification of textured forms, a rate that reaches 50% when the discrimination of these textures is effected from a global fractal dimension. In order to obtain the best results, textures are classified by using a number of parameters: fractal dimensions, wavelet coefficients and various entropy indicators.

Another more effective approach in the identification of textures is to calculate not one but several fractal dimensions over the eight or 16 directions of an image. These dimensions are then reported on a wind-rose graph and the figures obtained are estimated from multiple parameters. Each one contains information on texture, and it is therefore possible to classify these textures. We do not know of any application in geography that uses this approach, for example in order to classify urban textures, as this method currently remains confined to image processing.

#### 9.2.1.3. *Compressing the information*

Third, the number and size of images forces an information compression step. By reflecting again on the astrophysics example, images produced in the 1980s were generally 512 pixels squared, whereas images nowadays have increased to 16,000 pixels on each side. In remote sensing, the Ikonos satellite obtains similar-sized images. Wavelet formalism is responding very effectively to this kind of concern, and has adopted the JPEG 2000 format.

These compression techniques have inspired a lot of research at the great astronomical and satellite image records centers. Information fractal dimensions then help in the appreciation of compression quality. By comparing the Zipf log-log graphs of an original and a compressed image [CAR 04], it is possible to evaluate the distortion between these two images and therefore the compression quality. Some fractal coder-decoders are now on the market.

#### 9.2.1.4. *Creating realistic virtual images*

Finally, two-dimensional fractal simulation is an inexhaustible source of virtual images. They are omnipresent in digital games. They are currently pervading cinema screens, following on from the celebrated film *Star Wars*. They are responsible for creating new reliefs, exotic vegetation and worlds populated by imaginary creatures. All of these images are built using iterative function systems, which are becoming increasingly “realistic” thanks to developments in image processors installed in all personal computers. Classic software such as *Photoshop* proposes fractal effects.

Image fusion is easier using wavelet and fractal techniques. The interest in them within geography is growing. It is becoming very easy to integrate information issued from sources with different resolutions into a single image. A wavelet decomposition is performed on both images being merged. Wavelet coefficient couplings are then chosen. The inverse transform then provides a new fused image. Examples of this type of fusion can be found in the help files within *Mathematica*.

For geographers, this type of approach enables the recreation of original information. First, it is possible to modify the resolution of old satellite images by merging them with recent images with higher resolutions. Second, it is easier to follow the evolution of urbanization within a planning context by proceeding to such fusion with different types of images.

Unfortunately the fractal and multifractal approaches associated with wavelet decomposition are all too rarely used in geographical remote sensing. The most frequent applications refer to meteorological images [GRA 03]. On radar images it is possible to make out the high rainfall zones within a vast cloud-covered area, and to therefore to alert populations of an imminent flood risk.

Other applications are available, with a theoretical objective in most cases, such as better classification of ecological strata or landscapes, but without any practical application. T. Iftene *et al.* [IFT 04] have, however, demonstrated the superiority of this approach on techniques derived from the co-occurrence matrix for the differentiation of types of occupation of mostly agricultural space from Spot images.

In general, as emphasized by J. Lévy Véhel at the start of the fractal era, fractal and multifractal formalism was deployed in order to identify auto-similar objects. Nowadays this approach has become more operational. It is used on all sorts of phenomena to reveal certain *a priori* hidden properties.

### 9.2.2. *Some little-used fractal approaches using a GIS*

Although it is very frequently used in image processing, and much less in remote sensing, fractal formalism is still barely present in GIS. Admittedly some educational raster GIS systems, such as Landserf or Idrisi, calculate box-counting fractal dimensions; however, the limitations of these techniques have made them unsuitable for examining natural forms or those created by human societies.

In a raster GIS, fractal dimensions are calculated for evolving window sizes, and so it is therefore easy to study the multiscale change in fractal dimension by producing successive scans. A GIS therefore enables elementary multiscale image processing to be produced.

Some more innovative attempts have been made using the *ArcView* software, but these remain exceptional cases, as they require plug-ins to be programmed. The fractal dimensions calculated using GIS functionality are generally simple and refer to the area-to-perimeter ratio of previously identified objects [CHE 01]. In a recent article, M.F. Goodchild [GOO 09] tried to explain the rarity of these approaches. According to Goodchild, a vector GIS has little consideration for the multiscale characteristics of phenomena, apart from networks.

## 9.3. Fractals, optimization and land management

Fractals are an effective optimization tool. Numerous studies have demonstrated that a fractal object is an optimized form. For instance, the lungs occupy an enormous surface area, around 100 m<sup>2</sup>, within a limited volume. Replacing a straight line with a broken one makes the line longer. Admittedly, the traveling salesman's issue is that he aims to be the exception to the rule, but in many cases making use of a greater area or a longer line constitutes a double gain in terms of economy of space if the activity concerned is not a desirable one, but also in financial terms. There are many examples of the application of this principle.

In environmental management, for instance, fractals reduce the impact of vibrations. Through their configuration, fractal discontinuities are good shock absorbers. This law has been applied in physics and in engineering sciences. Thus, in order to reduce the plague of urban noise, fractal walls are now installed along certain axes. It should be possible to obtain similar, although less effective, results by anticipating the establishment of plant forms with spaces favoring the creation of fractal discontinuities.

This technique, which aims to absorb a negative impact, is already being used in coastal development. Beaches are being saved with the help of materials, but these

efforts are often negated by winter storms. Much more effective is the creation of fractal rocky outcroppings of embedded blocks of different sizes. These break the monotony of the coastline and thus blunt the effects of erosion, even if part of the force is sometimes displaced onto a neighboring section of the coastline. Research into fractals shows that it is advisable to make use of these outcroppings and other technical solutions in order to obtain a coastline whose fractal dimension is equal to 1.33, which leads to a dynamic equilibrium and to the coastline being maintained.

On the subject of environmental management, and more specifically biodiversity, the fractal paradigm is still a source of inspiration. It has been demonstrated that the fragmentation of a space, such as a forest, is detrimental to the existence of certain species; however, for small species, fragmentation of the same land may increase the niche effect and therefore favor their being planted and therefore their development or preservation. An optimal biosphere reserve would therefore be associated with small-scale homogeneous spaces for the large mammals, but low-level heterogeneous and fractal spaces so that smaller animals can survive. The fragmentation of an environment, so often criticized, has only negative effects.

Fractality has even become a decision-making tool in land development, since fractals optimally occupy space. Thus, in a presentation to *Géopoint*, the fractal approach enabled E. Bailly [BAI 08] to conceive an optimal organization of harbors. In this situation, the area occupied by the port cannot be modified.

In order to gain supplementary landing spaces, a fractalization of organization of the docks would be desirable. This is, in fact, the optimum solution – even if the size of pleasure boats and the navigation rules need to be taken into consideration. It should thus be possible to gain new spaces, which would be in high demand, within existing ports without having to build new ports that degrade the marine environment.

In a preliminary article, J. Cavahès [CAV 02] provided some general indications on this contribution of the fractal paradigm with regards to good urban governance. This idea is now being applied more operationally by Peking University geographers in order to optimize town organization [CHE 09]. They have compared the fractal dimensions of population and land utilization and – from this comparison – have drawn up rules for promoting quality urban development. For example, no fractal dimension of a type of land utilization – whether relating to industry or commercial space – should exceed the fractal dimension of the town.

In order to maximize the economic benefit of an urban collective, priority must be given to activities with the lowest fractal dimension. The Peking University

geographers therefore analyzed the town of Zhengzhou based on these rules and proposed a number of development solutions.

This type of approach is also recommended in order to improve the efficiency of transportation in general. C. Genre-Grandpierre [GEN 99] showed that the fractal dimensions from the distribution of bus stops were not optimal relative to the fractality of human densities in the Besançon agglomeration. These results needed to be converted into recommendations so that new bus stops could be localized. More recently, Besançon geographers have given credence to this field of research. Their MUP-city software should be an important decision-making tool for urbanists in research into land gain.

Even the terrorist threat is being combated with behavior inspired by fractal logic. For example, the optimum identification of hostages should be performed using a Lévy model, the efficiency of which has been highlighted – and occasionally doubted, with regards to research into nutrition. Rather than following a systematic and complete grid, it would appear to be more effective to proceed according to a Lévy movement. Quite clearly, this type of behavior would not be the same for identifying hostages in a desert, or in the heart of a global metropolis.

#### **9.4. Fractal beauty and landscaping**

For most geographers, beauty does not refer to science, but to art, even if our leaders have assimilated regional geography into an art form. Landscape development is therefore considered to be a functional, even aesthetic, practice. However, without intending to close this debate, the fractal approach results in some enriching information.

When a group of people are asked to classify landscape images according to their beauty, responses show that regular, homogeneous landscapes and highly irregular landscapes are never picked out as being the most beautiful. People who are constantly questioned attribute great beauty to landscapes with intermediate irregularity. This mix of regularity, which is calm and secure, and irregularity, which is synonymous with anomaly and surprise, can be measured using a Hurst coefficient or a fractal dimension.

Reliefs simulated by fractal techniques illustrate this assessment. A Hurst coefficient that is too close to 0 or 1 gives landscapes that are considered to be unreal, too irregular and too dull, respectively, and little admired. Beauty falls between these two extremes.

These ideas have been successfully tested in order to compare the outlines of great American towns [STA 02]. Highly planned towns with regular outlines, such as Salt Lake City, are perceived to be less beautiful than towns that have developed without any order being imposed.

Certain other factors affect this perception, in particular the alignments that make up the structure, the repetition of textures or the combinations of colors within textures. However, these techniques still play on the auto-similarity principle, as shown in Dali's painting *The Face of War*, the structure of which resembles a Cantor dust with a fractal dimension equal to 0.75. Similarly, some of Pollock's works were produced from movements resembling a Lévy flight. In this regard, R. P. Taylor [TAY 99] coined the term *fractal expressionism*.

A number of artists from the fractal school are starting to experiment with aesthetics and there are a number of interesting websites devoted to aesthetic research. Thus, random music is too monotonous and predictable when its distribution is Brownian. Some software packages, such as *Arti-Fract*, have been developed by the research team at INRIA to explore two-dimensional fractal space, particularly its visual qualities.

C. Redies *et al.* [RED 07] showed that natural landscapes and landscape paintings have fractal dimensions that are roughly equal, but very different to scenes from everyday life, such as that of an office. Indeed, the repetition of the same pattern over different levels is an artistic technique shared by all civilizations, as demonstrated in Persian carpet art, Arab friezes, Russian dolls and Gothic cathedrals. Fractal art existed before Mandelbrot dictated its geometry. The Apollonius figures, which are considered to be the first fractals, date from antiquity.

Some of these artistic rules would merit being identified and retained in landscape studies, where new laws relating to local development are being imposed across Europe.

## 9.5. Conclusion

The fractal paradigm teaches us that it is futile to want to manage development on one or two scales. Multiple ranges of spatiotemporal and physical, but also social and cultural scales should be considered. This principle is now accepted in hydrology for river management and deserves to be applied to all environmental and town management. Apart from trends that focus attention on local ecodevelopment or globalization, it would be advisable to favor multiscale practices for all environmental development and management issues.

Most land management tools are monoscale. This is true of environmental management impact studies, which deal with the environment at a highly localized project level, i.e. at a microscale, and, in all countries, if a coherence is researched, it is rarely found between local urban plans and regional strategic plans. There have been some recent efforts to obtain this coherence, but they all too often result in stacks of administrative or political – and therefore bureaucratic – responsibilities.



## Conclusion

“The last thing one finds when putting a book together  
is knowing what to put at the start.”

Blaise Pascal

There is much to be learned from reading our little book. First, by defining a fractal dimension, B. Mandelbrot opened the door to a new scientific paradigm. From a simple tool, a generation of researchers have developed methods and built theories. These theories and approaches are shared among a vast community, traversing various disciplinary boundaries. They are transforming our apprehension of the world.

Unlike other forms of knowledge, art and faith, science is progressive. The known universe is no longer that of Aristotle or the Humanists. This progress is sometimes thematic when new variables are made available to researchers. Science then, by extension, develops. Human geography has thus benefited from the empirical contributions of social geography and cultural approaches.

This scientific progress also comes from new techniques and methods. A simple standard deviation gives another view on the diversity provided by successive monographic studies. Mathematics is not simply to do with number crunching; it constitutes a way of thinking. In this way, it has an essentially qualitative component.

Finally, and especially important to note, the highest form of scientific progress is without doubt founded on new knowledge that renews or replaces prior knowledge. The fractal paradigm emphasizes these diverse forms of scientific development. Better still, despite being relatively new, it already offers practical applications in land-use development and environmental management.

### **C.1. Some tools and methods for quantifying and qualifying multiscale coarseness and irregularity**

On a technical level, the fractal paradigm offers an array of well-rounded tools to geographers:

- fractal dimensions;
- co-dimensions;
- lacunarities;
- semifractals;
- parabolic fractals; and
- multifractal spectrums.

Although we are presenting most of these techniques in this book, we do not claim this list of techniques to be exhaustive. For example, statisticians are still investigating the best way of measuring confidence limits in multifractal spectrums. Every year, new articles are devoted to these algorithms, notably those relating to the so-called bootstrap technique. Furthermore, many users have compared the merits of these tools against geostatistical techniques and occasionally the range of tools employed in image processing. Most of these comparisons apply to urban spread and are produced using *FRAGSTATS* software. Fractal techniques supplement these traditional tools well, but have hardly been used in the social sciences. Fractal techniques are occasionally successfully replacing them, even if the indicators derived from multiple categories of entropy are also very useful.

Against this technical profusion, geographers regularly encounter five difficulties that it is useful to mention here:

– First, the scaling parameter and fractal dimension should not be confused. Even various articles on rank-size distributions of urban hierarchies have not managed this.

– Second, it is important not to confuse multifractal with fractal scanning. A fractal scan provides the monofractal dimension from different parts of an object, for instance town quarters, while multifractal formalism, which deals with a single

object, reveals all the irregularities, intensity and discontinuity of that object at all scales. Having made this mistake ourselves, we have, as a result, noticed it in a great many subsequent works.

– A third difficulty deals with limits. By modifying the limits of an object, such as a town or cultural area, it makes sense that the fractal dimension changes. This change may help to identify discontinuities. These limits do not necessarily correspond to resolution, which is the lower limit of observation, and the extent or upper limit of the same observation. It is not mandatory for them to be fractal limits. Finally, it is advisable to avoid merging these various limits or discontinuities with crossover from changes in the scale invariance conditions, observed in the semi-fractal.

– Fourth, this work also places the emphasis on the low reliability of adjustments produced using the least squares method and through the interpretation of the determination coefficient. For relatively long series, comprised principally of positive values, this coefficient is often very high. The adjusted model is then described as pertinent, even if other modules are much better. Thus, it is clear that urban hierarchy follows a Pareto, rather than log-normal, distribution. This is verified for all urban hierarchy datasets, once the “best” model has been chosen, and cannot be tested on single log-normal adjustments.

– Fifth, in order to calculate the fractal dimension it is advisable to produce a preliminary array of tests so that the type of fractal pertaining to the geographical phenomenon being studied can be identified. Chapters 2, 3 and 4 amply demonstrate this. Our book shows that each type of fractal phenomenon possesses a “good” fractal dimension. Too many geographers apply calculation techniques that are valid for an auto-similar fractal, but unusable for self-affine fractals, even though they are more commonplace. This error appears to be quite widespread in geographical studies. Finally, other biases, which are described in this book, can be used when multifractal tools are available.

In the face of these problems, many geographers have been able to renounce and turn away from fractal and multifractal formalism, but they are wrong in doing this. This type of formalism is the most effective one for investigating irregularities, differences and discontinuities, not anecdotally but by identification across all scales. If some researchers and even more politicians are drawing the wrong conclusions from the correct data, then they should be able to think correctly based on the statistical results involving some bias.

On a methodological level, generalization of the fractal approach is still more beneficial. At a base level, it is a simple description of auto-similar forms, which are commonplace in mathematics but exceptional in nature and in the social world. The fractal approach therefore gains from self-affine sets. Then, from forms it applies to

processes, represented by chronicle series and frequential sequences. Finally, it renews the set approach provided by measurement and therefore all fields of geography. The fractal approach and multifractal methods are the glasses that lead to another view of the world.

### **C.2. A recap on geographical irregularities and disparities**

In spite of these technical and methodological contributions, geographers and social science specialists remain wary of this approach. It is easy to explain this cautiousness towards fractal formalism. In most major theories, science emphasizes the average phenomenon, which is no doubt a throwback to the gradualist positions of Lamarck and Darwin; nature did not make leaps. The evolution of species occurs slowly and regularly over time and space. In geography, this principle has been dominant in climatic geomorphology and rural geography for a long time. Landscapes develop slowly.

Although fractality places the emphasis on irregularities, coarseness, differences and territorial and temporal disparities, it is not without recourse to punctuated equilibrium theory. It offers us a new view of the world, where crashes are no longer exterior or exogenous, but are provoked by internal fluctuations. This is not a return to the old Cuvier catastrophism, but a description and explanation of the variability and irregularities of the universe across the scales.

A new interpretation of most geographical phenomena results from this approach. The majority of fractal objects and mechanisms possess non-properties. They effectively have no characteristic scale, not Gaussian, stationary or linear. Most importantly, there are two forms of geography relating to Gaussian or exponential phenomena and geography relating to the power laws. The former is suitable for describing a world with low variability, where the concept of an average phenomenon makes sense. The latter describes and explains a world where variability is high. When, in experimental data, the mean and variability are determined by high values, fractality becomes the rule.

From this assessment, new explanatory models are modifying our disciplinary, geographical, economical and sociological models. Let us take a look at a few examples. In physical geography, the fractal perturbation model, which is generated by turbulence in low-pressure areas, should replace the Norwegian model. While our writings have gone unheeded up to now, geographers should certainly accept the relevance of this new model based on studies conducted by meteorologists, who have all reached the same conclusion: pluviometric fields and multiscale are multifractals created from fractal forms as a result of turbulence.

This fractality also characterizes most natural hazards, such as earthquakes, volcanoes, cyclones and tempests, and more generally the majority of risks. Countless studies conducted in the various branches of the geosciences demonstrate this. The geography of risk geography is a fractal one.

In human geography, similar knowledge is making certain arguments questionable, for example those concerning the theme of center-periphery in population studies. Here it is clear that the center-periphery model, which is debatable for a given spatial level, is still relevant when considered in a multiscale context. Centers and peripheries fit into all territorial levels.

Yet all this new knowledge is missing, not only from academic textbooks – where analysis in terms of means and deviations from these means, which are a throwback to the Gaussian paradigm, are prevalent – but they are also unfortunately ignored in university textbooks. Sometimes these textbooks retain one of the two dominant characteristics of fractality, mostly the presence of extreme values. This is the case, for example, in urban hierarchy analysis. All textbooks dealing with the French urban network emphasize the primatial effect; the role of Paris. Despite this, scale invariance, which characterizes all of the rank-size distributions of these networks, is not mentioned.

In addition to the renewal of disciplinary explanations, fractality constitutes an entry point into the understanding and application of complexity theories in social sciences. As fractality extends beyond the field of geography, the answer is to research these complexity theories. These theories do not get a good press, since their name appears to glorify complexity, despite the fact that they are best explained by the iteration of relatively simple mechanisms, i.e. fractals.

These theories do, however, bring about a new understanding of phenomena, also sometimes explained using disciplinary theories. For example, when birds are in flocks, antelopes are in herds or sardines are in shoals, this is traditionally understood to be their best defense against predators. With this disciplinary explanation, complexity theories favor the production of these dynamic phenomena using three simple rules that are “blindly” applied by each of the participants, making these animals brilliant specialists in optimization.

In urban geography, the disciplinary explanations followed by economists attribute rank-size distributions to exogenous development, which advances Gibrat's principle, or endogenous development, which is based on the production and then diffusion of innovation. Studies conducted by specialists in urban geography favor the former interpretation before building an evolutive theory inspired by the latter, which was promoted by Schumpeter. Moreover, these same geographers appealed to spatial economists to explain the spatial organization of polygonal urban networks.

Two schools of thought are required to explain the two facets of the same phenomenon – the hierarchical urban network.

We have been able to show that critical self-organized systems theory provides a satisfactory explanation for the double fractality of urban hierarchies, which are represented by rank-size distributions and corresponding spatial distribution. What was opposed by P. Claval, however, is now back in favor. Intermittence is at the heart of the mechanism forming these networks. Phases of network stability are interspersed with phases of instability. Networks are thus dynamically organized in space and time, passing from the local networks of antiquity to the global villages in place during the 21<sup>st</sup> Century.

Of all the many complexity theories, our attention has been focused on four in this book. Misplaced pride would lead us to consider our own, derived from numerous diverse models, as being the best. A more effective solution consists of investigating their common point. All of these theories share a common mechanism: movement, whether from normal or abnormal diffusion, from advection–convection, or from turbulence. All movement is formalized by an equation that is invariant through scale transformation. This is true of Brownian diffusion, as with all other equations that formalize movement, such as the Navier-Stokes equation which modelizes multifractal turbulence in air, oceans or even panic movements in stadiums. As soon as geographers introduce movement into their reasoning, fractality increases. The great painters interested in movement no doubt felt this unconsciously. As proof, we have Katsushika Hokusai's great wave, Leonardo da Vinci's schemas and Vincent Van Gogh's starry night.

In conclusion, classical geographers studied irregularity on a case-by-case basis, by composing monographs. It was then a question of causing uniqueness: the specificity of a slope, forest, town or region thanks to artistic writing, which is evocative of this uniqueness. The quantitative geographers of the 1960s centered their attentions on the basic facts. Originality was restored through the study of simple or multiple simple regression models. The fractal paradigm reconciles geographers with the richness of the world. It tracks and highlights coarseness and the spatiotemporal irregularities of territories. Irregularity is no longer an accident, but is becoming the subject of scientific research. It possesses or illustrates an order that the fractal approach helps to describe and explain.

Better still, coupled with Fourier, entropy or wavelet decomposition models, the fractal paradigm enables multiscale interactions to be tackled scientifically.

### C.3. A paradigm that gives rise to new land-use management practices

The fractal and multiscale approach is not simply useful for understanding geographical phenomena. It needs to become the rule in order to ensure good management, optimal development of territories or best environmental management. There is a better understanding of the failure of planned economies. In spite of their efforts, they never manage to establish any multiscale coherence; they virtually all fail on this contradiction. Quite clearly, we should not conclude from this that all planning or regulation should be prohibited. The example of the medical or technological handling of epidemics proves the opposite. Random strategies without any regulation also lead to failures. In other words, it is not due to the failure of socialism that “*laissez-faire*” or liberalism will be a permanent success. The invisible hand of A. Smith, if it ever existed, would not be a guarantee of success.

More prosaically, faced with the necessity to implement multiscale land and environment management, we should encourage the creation of multiscale tools. In all states of the world, the zoning plans, the city plans and the metropolitan plans should be replaced by a single, multiscale urbanization plan. Certainly current legislation, where regulation is attached to a fairly delimited area, is opposed to this. The fractal approach may therefore provide, if not complete solutions, then at least strategies for overturning this legal constraint. We must not forget that urban zoning has only reinforced, if not created, segregation phenomena.

Despite the undeniable progress that brought about the fractal approach, many technical, methodological and theoretical questions remain unanswered. Geographers are not necessarily best qualified to bring about technical or methodological solutions. They need to be inspired by the works of statisticians and information technology professionals. On the other hand, their contribution is important in the theoretical domain as they are able to make sense of formal complexity theories, which is without doubt most useful in the comprehension of world fractality, given our current levels of understanding.

Our world has changed greatly. Beyond the epiphenomena that create newspaper headlines, three properties have influenced this change.

First, societies and people are becoming aware that they are living in a finite world. Admittedly, science will push this to its limits: petroleum coke, with its ecological consequences will replace natural oil in a few centuries. Despite precedence, if not from their work then at least from their position, geographers have been well placed to respond to questions arising from this perception of a finite world. However, ecologicist ideologists have supplanted scientists and geographers as ecologists.

Second, the world is asserting itself as a set of interconnected networks. This is not a new concept. The town network has been identified since the works of M. Castells [CAS 96]. With the exception of some brilliant spatial analysis specialists, however, geographers have invested little in this field of knowledge. Some sociologists have managed to occupy this neglected space.

Third, the world is undergoing a contradictory evolution. Globalization and internationalism go hand-in-hand with fragmentation, differentiation and the continuous creation of disparities. Cycles in innovation that are sweeping across the Earth can only create homogeneity at the end of a cycle, when all living beings are using the same products, whether this is of material benefit or some cultural representation. As innovation never stops, differentiation and fragmentation are intermingled on all scales. It is this fundamental contradiction that justifies a fractal geography, which brings to light interactions between levels; a fractal geography capable of describing, understanding and explaining the macroevolution of the world system.

## Appendices

### A.1. Preliminary thoughts on fractal analysis software

There are a number of software packages available for the analysis of forms and fractal processes, which can be broken down into three categories.

The first category includes image processing or Raster GIS (Geographical Informations Systems) software, such as *Fovea*, *Idrisi*, *ENVI*, *Landserf* and various NIH (National Insitut Health) Image plugins, particularly *Fraclac*. These software packages have two main features. First, they enable a number of other complementary treatments, notably in the area of geostatistics. Second, they virtually all incorporate fractal scanning algorithms that enable an image to be cut into sections and then the fractal dimensions of each subimage resulting from this decomposition to be calculated. It is easy, using *Landserf*, for example, to observe the variations in these fractal dimensions based on the size of successive decompositions.

The second category of software packages has been produced by teams of geographers. The best known in French geography is *Fractalyse*, which was developed by the ThéMA (*Théoriser et Modéliser pour Aménager*) team in Besançon, headed by G. Vuidel. Recent versions are written in *MatLab* language. A large number of options are available, inspired by problems from urban geography. *Fractalyse* is a significant upgrade from the first software packages, which were written in *Fortran* by American geographers.

The third category, containing the vast majority of these packages that are relevant to the treatment of auto-similar fractals, are less suitable for the analysis of self-affine fractals and still do not, as yet, offer modules for analyzing multifractal forms and processes, although the same would apply even if this gap had been bridged.

Furthermore, we would have hoped that adjustment techniques more relevant to log-log than least squares distributions could have been programmed.

These limitations have led researchers to create new software, mostly modules written in the C/C++ or MatLab programming languages, of which there are a significant number. In France there are three of particular interest:

- *LastWave*, which was built by E. Bacry at the École Polytechnique;
- *FracLab*, which was developed at INRIA under the leadership of J. Lévy Véhel; and
- various modules from the ENS (*Ecole Normale Supérieure*) in Lyon, notably those written by P. Abry and his student team.

These packages all require a solid information technology infrastructure.

Here we propose a final solution, developed around the *Mathematica* software package. There are several reasons for this choice. *Mathematica* enables virtually anything to be produced. Besides the writing of articles and books and the production and presentation of PowerPoint slideshows, it makes use of mathematical tools for handling arrays, developing macro- and microsimulation models, image and map processing and producing all types of statistical analyses. As with *MatLab*, hundreds of studies have been devoted to it and there may be more than 7,000 demonstration projects, some of which are true research tools. The only downside appears to be that learning it requires assiduous practice.

## A.2. Instructions for the following programs

All of the programs listed below can be used directly. They are comprised of an import and data-reading module and processing and graphic and/or digital reporting modules. The data-reading module can be adapted by the user based on the nature and format of the data to be examined.

The processing module avoids the stacking of numerous instructions within a single line of code. This good programming practice in *Mathematica* has the downside of being barely readable for the novice. We have therefore preferred to write our programs instruction-by-instruction, which may be less optimal but is more comprehensible.

Finally, for each digital or graphical output instruction, there are numerous options. They may enable users to obtain magnificent graphs, but, again, the programs proposed only produce simple black and white graphs suitable for later

publication. These graphs can easily be completed, selected and saved using pallets and menus from *Mathematica*.

The programs are grouped into four categories. The first category of programs visually process unique series, which may be chronicle series or frequencies, but also images. The programs provide Pareto and Zipf graphs of these. The second category comprises programs that calculate the fractal dimensions of the series and the frequential distributions. The programs in the third category apply to satellite images, maps and to geography in general. Finally, some programs are devoted to multifractal formalism. A brief comment summarizes some notes necessary for the best application of each program.

These programs were tested on an iMac, with datasets comprising between 256 and 2,048 values. For image processing these values corresponded to image sizes measured in numbers of pixels. In some cases, some small modifications were subsequently required, notably for programs using Fourier decomposition, if the observations varied significantly.

### A.3. Software programs for the visual approach of a satellite or cartographic series or image

We are proposing three software programs: one for a series that may be a chronicle series or series of discrete data; a second for a square image; and a third that produces the same instructions, but on subimages of an image. It is a visual scanning program.

PROGRAM A.1.—

```
(*Graphic visualization and processing of a series with production of Pareto and
Zipf graphs*)
(*Import of data saved in Mathematica format*)
ClearAll["Global`*"]
d = Import[SystemDialogInput["FileOpen"]];
d = Flatten[d];
n1 = Length[d];
(*Data sorting*)
d1 = Sort[d, Greater];
d2 = Table[i, {i, 1, n1}];
d3 = Riffle[d2, d1];
d6 = Partition[d3, 2];
(*Graphic representations*)
Print["Length of series = ", n1]
```

```

Print["Pareto graph"]
a1 = ListLogLogPlot[{d6}, ImageSize -> {300, 200}, Filling -> Axis,
  Joined -> True, PlotLabel -> "Log-log", AxesLabel -> {"Rank", "Size"},
  PlotRange -> All, PlotStyle -> {Black}];
a2 = ListLogPlot[{d6}, ImageSize -> {300, 200}, Filling -> Axis,
  Joined -> True, PlotLabel -> "Log-arithmetic",
  AxesLabel -> {"Rank", "Size"}, PlotRange -> All, PlotStyle -> {Black}];
a3 = ListPlot[{d6}, ImageSize -> {300, 200}, Filling -> Axis, Joined -> True,
  PlotLabel -> "Arithmetic", AxesLabel -> {"Rank", "Size"},
  PlotRange -> All, PlotStyle -> {Black}];
GraphicsColumn[{a1, a2, a3}, Frame -> All]
Print[]
Print["Zipf graphs"]
d3 = Riffle[d1, d2];
d6 = Partition[d3, 2];
a1 = ListLogLogPlot[{d6}, ImageSize -> {300, 200}, Filling -> Axis,
  Joined -> True, PlotLabel -> "Log-log", AxesLabel -> {"Size", "Rank"},
  PlotRange -> All, PlotStyle -> {Black}];
a2 = ListLogPlot[{d6}, ImageSize -> {300, 200}, Filling -> Axis,
  Joined -> True, PlotLabel -> "Log-arithmetic",
  AxesLabel -> {"Size", "Rank"}, PlotRange -> All, PlotStyle -> {Black}];
a3 = ListPlot[{d6}, ImageSize -> {300, 200}, Filling -> Axis, Joined -> True,
  PlotLabel -> "Arithmetic", AxesLabel -> {"Size", "Rank"},
  PlotRange -> All, PlotStyle -> {Black}];
GraphicsColumn[{a1, a2, a3}, Frame -> All]

```

COMMENT A.1.— The graphs use arithmetic, log-arithmetic and log-log coordinates. A power law is illustrated by a straight line on the log-log graph, whereas the exponential distribution is also a straight line, but on a log-arithmetic graph.

PROGRAM A.2.—

```

(*Graphic visualization and processing of an image*)
(*Image import and display*)
ClearAll["Global`*"]
donimage = Import[SystemDialogInput["FileOpen"]];
Show[donimage]
(*Calculation of two means of rows and columns*)
d = N[ImageData[donimage, "Byte"]];
d0 = Mean[d];
d00 = Mean[Transpose[d]];

```

```

n1 = Length[d];
(*Pareto and Zipf models by row*)
Print["Number of pixels per row = "]
Print["Graphs of row averages"]
d1 = Sort[d0, Greater];
dx1 = d1;
d2 = Table[i, {i, 1, n1}];
d3 = Riffle[d2, d1];
d6 = Partition[d3, 2];
Print[]
Print["Pareto graphs for average rows"]
a1 = ListLogLogPlot[{d6}, ImageSize -> {300, 200}, Filling -> Axis,
  Joined -> True, PlotLabel -> "Log-log", AxesLabel -> {"Rank", "Size"},
  PlotRange -> All, PlotStyle -> {Black}];
a2 = ListLogPlot[{d6}, ImageSize -> {300, 200}, Filling -> Axis,
  Joined -> True, PlotLabel -> "Log-arithmetic",
  AxesLabel -> {"Rank", "Size"}, PlotRange -> All, PlotStyle -> {Black}];
a3 = ListPlot[{d6}, ImageSize -> {300, 200}, Filling -> Axis, Joined -> True,
  PlotLabel -> "Arithmetic", AxesLabel -> {"Rank", "Size"},
  PlotRange -> All, PlotStyle -> {Black}];
GraphicsColumn[{a1, a2, a3}, Frame -> All]
Clear[d3, d6, a1, a2, a3];
Print["Zipf graphs for average rows"]
d3 = Riffle[d1, d2];
d6 = Partition[d3, 2];
a1 = ListLogLogPlot[{d6}, ImageSize -> {300, 200}, Filling -> Axis,
  Joined -> True, PlotLabel -> "Log-log", AxesLabel -> {"Size", "Rank"},
  PlotRange -> All, PlotStyle -> {Black}];
a2 = ListLogPlot[{d6}, ImageSize -> {300, 200}, Filling -> Axis,
  Joined -> True, PlotLabel -> "Log-arithmetic",
  AxesLabel -> {"Size", "Rank"}, PlotRange -> All, PlotStyle -> {Black}];
a3 = ListPlot[{d6}, ImageSize -> {300, 200}, Filling -> Axis, Joined -> True,
  PlotLabel -> "Arithmetic", AxesLabel -> {"Size", "Rank"},
  PlotRange -> All, PlotStyle -> {Black}];
GraphicsColumn[{a1, a2, a3}, Frame -> All]
Clear[d3, d6, a1, a2, a3];
(*Pareto and Zipf models by column*)
Print["Pareto graphs for average columns"]
d1 = Sort[d00, Greater];
dxc = d1;
d2 = Table[i, {i, 1, n1}];

```

```

d3 = Riffle[d2, d1];
d6 = Partition[d3, 2];
Print[]
a1 = ListLogLogPlot[{d6}, ImageSize -> {300, 200}, Filling -> Axis,
  Joined -> True, PlotLabel -> "Log-log", AxesLabel -> {"Rank", "Size"},
  PlotRange -> All, PlotStyle -> {Black}];
a2 = ListLogPlot[{d6}, ImageSize -> {300, 200}, Filling -> Axis,
  Joined -> True, PlotLabel -> "Log-arithmetic",
  AxesLabel -> {"Rank", "Size"}, PlotRange -> All, PlotStyle -> {Black}];
a3 = ListPlot[{d6}, ImageSize -> {300, 200}, Filling -> Axis, Joined -> True,
  PlotLabel -> "Arithmetic", AxesLabel -> {"Rank", "Size"},
  PlotRange -> All, PlotStyle -> {Black}];
GraphicsColumn[{a1, a2, a3}, Frame -> All]
Clear[d3, d6, a1, a2, a3];
Print[]
Print["Zipf graphs for average columns"]
d3 = Riffle[d1, d2];
d6 = Partition[d3, 2];
a1 = ListLogLogPlot[{d6}, ImageSize -> {300, 200}, Filling -> Axis,
  Joined -> True, PlotLabel -> "Log-log", AxesLabel -> {"Size", "Rank"},
  PlotRange -> All, PlotStyle -> {Black}];
a2 = ListLogPlot[{d6}, ImageSize -> {300, 200}, Filling -> Axis,
  Joined -> True, PlotLabel -> "Log-arithmetic",
  AxesLabel -> {"Size", "Rank"}, PlotRange -> All, PlotStyle -> {Black}];
a3 = ListPlot[{d6}, ImageSize -> {300, 200}, Filling -> Axis, Joined -> True,
  PlotLabel -> "Arithmetic", AxesLabel -> {"Size", "Rank"},
  PlotRange -> All, PlotStyle -> {Black}];
GraphicsColumn[{a1, a2, a3}, Frame -> All]
dxlc = Riffle[dxl, dxc];
dxlc6 = Partition[dxlc, 2];
maxdc = Max[dxlc6];
Print[]
Print["Bigraph of average column and row values"]
a1 = ListLinePlot[Range[maxdc]];
a2 = ListPlot[{dxlc6}, AxesLabel -> {"rows", "columns"},
  GridLines -> Automatic];
Show[a2, a1]

```

COMMENT A.2.— This program proposes preliminary visual processing of data from a square image or map. It gives the same results as the previous program. These graphs are only plotted for average row and column calculations. A bigraph

compares the visual organization of these average rows and columns. For a rectangular image, small adjustments would be necessary. This program can be improved by the creation of a module, instead of repeating the same instructions.

PROGRAM A.3.—

```
(*Zipf and Pareto graphs of sub-images*)
(*Import and visualization of image of 64, 128, 256,... pixels per side*)
ClearAll["Global`*"]
im = Import[SystemDialogInput["FileOpen"]];
Show[im]
(*Calculation of image size and conversion into digital data *)
n1 = ImageDimensions[im];
n1 = n1[[1]];
imad = ImageData[im, "Byte"];
d = Flatten[imad];
d = Sort[d, Greater];
nd = Length[d];
(*rank-size distribution for the whole image*)
Print["Rank-size distribution for the whole image"]
d4 = Table[i, {i, 1, nd}];
d5 = Riffle[d4, d];
d6 = Partition[d5, 2];
im1 = ListLinePlot[d6, AxesLabel -> {"Rank", "Size"},
  PlotLabel -> "Arithmetic", PlotRange -> All, Filling -> Axis];
d7 = Log[d6];
im2 = ListLinePlot[d7, AxesLabel -> {"Rank", "Size"},
  PlotLabel -> "Arithmetic", PlotRange -> All, Filling -> Axis];
GraphicsRow[{im1, im2}, ImageSize -> {400, 300}, Frame -> All]
(*choose partition size*)
Print[]
n2 = N[ChoiceDialog[
  "Choose the sub-image size in pixels", {16 -> "16", 32 -> "32",
    64 -> "64", 128 -> "128", 256 -> "256", 512 -> "512"}]];
n2 = ToExpression[n2];
Print["Size of sub-images = ", n2]
n3 = n1/n2;
Print[]
Print["Rank-size distribution of ", n3^2, " sub-images"]
imagepart4 = Flatten[ImagePartition[im, n2]];
d2 = Map[ImageData[#, "Byte"] &, imagepart4] // N;
d = Flatten[d2];
```

```

dd = Partition[d, {n2*n2}];
(*Rank-size distribution for each sub-image*)
f[n_] := Module[{xx = dd, dx, d10, imadc, nd, d4, d5, d6, im1, d7, im2},
  dx = dd[[n]];
  d10 = Partition[dx, n2];
  imadc = Image[d10, "Byte"];
  dx = Sort[dx, Greater];
  nd = Length[dx];
  d4 = Table[i, {i, 1, nd}];
  d5 = Riffle[d4, dx];
  d6 = Partition[d5, 2];
  im1 = ListLinePlot[d6, AxesLabel -> {"Rank", "Size"},
    PlotLabel -> "Arithmetic", PlotRange -> All, Filling -> Axis];
  d7 = Log[d6];
  im2 = ListLinePlot[d7, AxesLabel -> {"Rank", "Size"},
    PlotLabel -> "logarithmic", PlotRange -> All, Filling -> Axis];
  GraphicsRow[{im1, im2}, ImageSize -> {400, 300}, Frame -> All]
]
Dq = Table[f[i], {i, 1, n3^2, 1}]

```

COMMENT A.3.— This program proposes visual scanning of subimages of an image that is  $2^n$  pixels squared. The size of the subimages is chosen by the user based on answers to questions posed by the program and must conform to the size of the original image. Thus, for an image 128 pixels squared, it should be possible to obtain four graphs for a partition into images 64 pixels squared, or 64 graphs if the partition provides images that are 16 pixels squared. This proliferation of graphs for more “realistic” images, for example satellite photographs that are 2,048 pixels squared, cannot be obtained by repeating the same instructions. The program therefore contains a module equivalent to a subroutine, which is called for each subimage. The last instruction determines the shape of the graphs for each subimage. Note the facility for condensing the iterations using *Mathematica* language.

#### A.4. Software programs for calculating fractal dimensions for a chronicle or frequency series

We are proposing six programs. The first two test various adjustments for a series arranged according to a rank-size distribution. These programs are applied to either discrete or continuous data. The following programs calculate a fractal dimension using a variogram, an entropy, a Fourier decomposition and finally a wavelet decomposition.

PROGRAM A.4.–

```
(*Choose an adjustment distribution for a discrete data series*)
(*Data import*)
ClearAll["Global`*"]
d = Import[SystemDialogInput["FileOpen"]];
d = Flatten[d];
n1 = Length[d];
Print["maximum value = ", m = Max[d]]
Print["minimum value = ", min = Min[d]]
Print["mean value = ", moy = Mean[d]]
Print["standard deviation = ", sd = StandardDeviation[d]]
Print["beta = ", beta = (sd* Sqrt[6])/π]
Print["alpha = ", alpha = moy - (0.5772*beta)]
(*Sorting of data*)
d = Sort[d, Greater];
d4 = Table[i, {i, 1, n1}];
d5 = (n1 - d4)/n1;
d2 = Riffle[d, d5];
d6 = Partition[d2, 2];
(*Zipf distribution adjustment*)
CDF[ZipfDistribution[k, α], x]
nlm = NonlinearModelFit[d6, CDF[ZipfDistribution[k, α], x], {k, α}, x]
vals = nlm["BestFitParameters"]
ZipfCDF = CDF[ZipfDistribution[k, α], x] /. %
Show[ListPlot[d6, AxesLabel -> {"towns", "cdf"}, ImageSize -> {300, 200},
Joined -> True, PlotRange -> All, PlotStyle -> {Black},
PlotLabel -> "Zipf distribution"]]
Print["Akaike criteria = ", akaike = nlm["AIC"]]
Print["Bayes criteria = ", Bayes = nlm["BIC"]]
Print["Determination coefficient = ", R2 = nlm["RSquared"]]
Clear[nlm, vals]
(*Benini distribution adjustment*)
CDF[BeniniDistribution[k, α, σ], x]
nlm = NonlinearModelFit[d6, CDF[BeniniDistribution[k, α, σ], x], {k, α, σ},
x]
vals = nlm["BestFitParameters"]
ZipfCDF = CDF[BeniniDistribution[k, α, σ], x] /. %
Show[ListPlot[d6, AxesLabel -> {"towns", "cdf"}, ImageSize -> {300, 200},
Joined -> True, PlotRange -> All, PlotStyle -> {Black},
PlotLabel -> "Benini distribution"]]
Print["Akaike criteria = ", akaike = nlm["AIC"]]
```

```

Print["Bayes criteria = ", Bayes = nlm["BIC"]]
Print["Determination coefficient = ", R2 = nlm["RSquared"]]
Clear[nlm, vals]
(*Log-Normal distribution adjustment*)
CDF[LogNormalDistribution[μ, σ], x]
nlm = NonlinearModelFit[d6, CDF[LogNormalDistribution[μ, σ], x], {μ, σ}, x]
vals = nlm["BestFitParameters"]
LogNormalCDF = CDF[LogNormalDistribution[μ, σ], x] /. %
Show[ListPlot[d6, AxesLabel -> {"town", "cdf"}, ImageSize -> {300, 200},
  Joined -> True, PlotRange -> All, PlotStyle -> {Black},
  PlotLabel -> "Log-normal distribution"]]
Print["Akaike criteria = ", akaike = nlm["AIC"]]
Print["Bayes criteria = ", Bayes = nlm["BIC"]]
Print["Determination coefficient = ", R2 = nlm["RSquared"]]
Clear[nlm, vals]
(*Parabolic fractal adjustment*)
nlm =
  NonlinearModelFit[d6, {a + b x + c x^2}, {a, b, c}, x]
vals = nlm["BestFitParameters"]
Show[ListPlot[d6, AxesLabel -> {"towns", "cdf"}, ImageSize -> {300, 200},
  Joined -> True, PlotRange -> All, PlotStyle -> {Black},
  PlotLabel -> "Parabolic Fractal"]]
Print["Akaike criteria = ", akaike = nlm["AIC"]]
Print["Bayes criteria = ", Bayes = nlm["BIC"]]
Print["Determination coefficient = ", R2 = nlm["RSquared"]]

```

COMMENT A.4.— This program, which was constructed from a tutorial developed by F. Maclachlan [MAC 06], is not optimized. It is formed of repetitive blocks that should have been included in one of the *demonstration projects*. Most importantly, *Mathematica 8.0.1*, which is the latest version available, enables this type of approach to be considerably simplified, since it offers new solutions. The advantage of program 5 below is that it is also compatible with version 7, which is still often used.

PROGRAM A.5.—

```

(*Choose an adjustment distribution from a continuous set of data*)
(*Data import*)
ClearAll["Global`*"]
d = Import[SystemDialogInput["FileOpen"]];
d = Flatten[d];
n1 = Length[d];

```

```

Print["maximum value = ", m = Max[d]]
Print["minimum value = ", min = Min[d]]
Print["mean value = ", moy = Mean[d]]
Print["standard deviation = ", sd = StandardDeviation[d]]
Print["beta = ", beta = (sd* Sqrt[6])/π]
Print["alpha = ", alpha = moy - (0.5772*beta)]
(*Calculation of data frequency*)
n2 = Input["Choose the frequential interval size"];
n2 = ToExpression[n2];
Print["Size of frequential intervals = ", n2]
freq = BinCounts[d, {0, m, n2}];
tot = Total[freq];
Print["Cumulative frequencies"]
freqcum = N[Accumulate[freq]/tot];
d4 = Table[i, {i, n2, m, n2}];
d5 = Riffle[d4, freqcum];
d6 = Partition[d5, 2];
(*Pareto I adjustment*)
CDF[ParetoDistribution[k, α], x]
nlm = NonlinearModelFit[d6, CDF[ParetoDistribution[k, α], x], {k, α}, x]
vals = nlm["BestFitParameters"]
ParetoCDF = CDF[ParetoDistribution[k, α], x] /. %
Show[ListPlot[d6, AxesLabel -> {"towns", "cdf"}, ImageSize -> {300, 200},
Joined -> True, PlotRange -> All, PlotStyle -> {Black},
PlotLabel -> "Pareto I distribution"]]
Print["Akaike criteria = ", akaike = nlm["AIC"]]
Print["Bayes criteria = ", Bayes = nlm["BIC"]]
Print["Determination coefficient = ", R2 = nlm["RSquared"]]
Clear[nlm, vals]
(*Pareto IV adjustment*)
CDF[ParetoDistribution[k, α, γ, μ], x]
nlm = NonlinearModelFit[d6,
CDF[ParetoDistribution[k, α, γ, μ], x], {k, α, γ, μ}, x]
vals = nlm["BestFitParameters"]
ParetoCDF = CDF[ParetoDistribution[k, α, γ, μ], x] /. %
Show[ListPlot[d6, AxesLabel -> {"towns", "cdf"}, ImageSize -> {300, 200},
Joined -> True, PlotRange -> All, PlotStyle -> {Black},
PlotLabel -> "Pareto IV distribution"]]
Print["Akaike criteria = ", akaike = nlm["AIC"]]
Print["Bayes criteria = ", Bayes = nlm["BIC"]]
Print["Determination coefficient = ", R2 = nlm["RSquared"]]

```

```

Clear[nlm, vals]
(*Power law adjustment*)
CDF[PowerDistribution[k,  $\alpha$ ], x]
nlm = NonlinearModelFit[d6, CDF[PowerDistribution[k,  $\alpha$ ], x], {k,  $\alpha$ }, x]
vals = nlm["BestFitParameters"]
PowerCDF = CDF[PowerDistribution[k,  $\alpha$ ], x] /. %
Show[ListPlot[d6, AxesLabel -> {"towns", "cdf"}, ImageSize -> {300, 200},
  Joined -> True, PlotRange -> All, PlotStyle -> {Black},
  PlotLabel -> "Power Law"]]
Print["Akaike criteria = ", akaike = nlm["AIC"]]
Print["Bayes criteria = ", Bayes = nlm["BIC"]]
Print["Determination coefficient = ", R2 = nlm["RSquared"]]
Clear[nlm, vals]
(*Lévy distribution adjustment*)
CDF[LevyDistribution[k,  $\alpha$ ], x]
nlm = NonlinearModelFit[d6, CDF[LevyDistribution[k,  $\alpha$ ], x], {k,  $\alpha$ }, x]
vals = nlm["BestFitParameters"]
LevyCDF = CDF[LevyDistribution[k,  $\alpha$ ], x] /. %
Show[ListPlot[d6, AxesLabel -> {"towns", "cdf"}, ImageSize -> {300, 200},
  Joined -> True, PlotRange -> All, PlotStyle -> {Black},
  PlotLabel -> "Lévy Distribution"]]
Print["Akaike criteria = ", akaike = nlm["AIC"]]
Print["Bayes criteria = ", Bayes = nlm["BIC"]]
Print["Determination coefficient = ", R2 = nlm["RSquared"]]
Clear[nlm, vals]
(*Log-Normal distribution adjustment*)
CDF[LogNormalDistribution[ $\mu$ ,  $\sigma$ ], x]
nlm = NonlinearModelFit[d6, CDF[LogNormalDistribution[ $\mu$ ,  $\sigma$ ], x], { $\mu$ ,  $\sigma$ }, x]
vals = nlm["BestFitParameters"]
LogNormalCDF = CDF[LogNormalDistribution[ $\mu$ ,  $\sigma$ ], x] /. %
Show[ListPlot[d6, AxesLabel -> {"towns", "cdf"}, ImageSize -> {300, 200},
  Joined -> True, PlotRange -> All, PlotStyle -> {Black},
  PlotLabel -> "Log-Normal Distribution"]]
Print["Akaike criteria = ", akaike = nlm["AIC"]]
Print["Bayes criteria = ", Bayes = nlm["BIC"]]
Print["Determination coefficient = ", R2 = nlm["RSquared"]]
Clear[nlm, vals]
(*Parabolic fractal adjustment*)
nlm = NonlinearModelFit[d6, {a + b x + c x^2}, {a, b, c}, x]
vals = nlm["BestFitParameters"]
Show[ListPlot[d6, AxesLabel -> {"towns", "cdf"}, ImageSize -> {300, 200},

```

```

Joined -> True, PlotRange -> All, PlotStyle -> {Black},
PlotLabel -> "Parabolic Fractal"]]]
Print["Akaike criteria = ", akaike = nlm["AIC"]]
Print["Bayes criteria = ", Bayes = nlm["BIC"]]
Print["Determination coefficient = ", R2 = nlm["RSquared"]]
(*EVD adjustment*)
CDF[ExtremeValueDistribution[ $\alpha$ ,  $\beta$ ], x]
nlm = NonlinearModelFit[d6, CDF[ExtremeValueDistribution[ $\alpha$ ,  $\beta$ ], x], { $\alpha$ ,  $\beta$ },
x]
vals = nlm["BestFitParameters"]
ExtremeValueDistributionCDF = CDF[ExtremeValueDistribution[ $\alpha$ ,  $\beta$ ], x] /. %
Show[ListPlot[d6, AxesLabel -> {"towns", "cdf"}, ImageSize -> {300, 200},
Joined -> True, PlotRange -> All, PlotStyle -> {Black},
PlotLabel -> "Extreme Value Distribution"]]]
Print["Akaike criteria = ", akaike = nlm["AIC"]]
Print["Bayes criteria = ", Bayes = nlm["BIC"]]
Print["Determination coefficient = ", R2 = nlm["RSquared"]]

```

COMMENT A.5.— This program presents identical characteristics to the previous one. Some additional instructions enable frequential classifications to be produced, the amplitude of which is chosen by the user. The program then enables cumulative frequencies or probabilities to be calculated. The extreme values distribution corresponds to Fréchet's distribution.

PROGRAM A.6.—

```

(*Calculation of the fractal dimension from variogram on a set of data *)
(*Data import *)
ClearAll["Global`*"]
d = Import[SystemDialogInput["FileOpen"]];
d = Flatten[d];
d = d - Mean[d];
n1 = Length[d];
n2 = Floor[n1/4];
(*Calculation from the variogram*)
Print[]
Print["Variogram"]
v1 = Variance[d];
Autocovariance[y_, dtemp_] := Return[Covariance[y, RotateLeft[y, dtemp]]]
d3 = Table[(v1 - Autocovariance[d, n1])/2, {n1, 0, n2}];
ListPlot[{d3}, ImageSize -> {500, 400}, Filling -> Axis,
AxesLabel -> {"dt", "Variogram"}, PlotStyle -> PointSize[Medium]]

```

```

d4 = d3 - RotateLeft[d3, 1];
ListPlot[{d4}, ImageSize -> {500, 400}, Filling -> Axis,
  AxesLabel -> {"dt", "difference"}, PlotStyle -> PointSize[Medium]]
n3 = DialogInput[
  DynamicModule[{name = ""}, Column[{
    "Choose the cut size",
    InputField[Dynamic[name], String],
    ChoiceButtons[{DialogReturn[name], DialogReturn[]}]
  }]]];
n3 = ToExpression[n3];
(*Calculation of the log and fractal dimension of the variogram*)
d5 = Table[i, {i, 1, n3 - 1, 1}];
d3 = Take[d3, {2, n3}];
d5a = Riffle[d5, d3];
d6 = Partition[d5a, 2];
d6 = Log[d6];
line = LinearModelFit[d6, {1, x}, x];
linen = Normal[line];
da = linen[[2, 1]];
df = N[(4 - Abs[da])/2];
Print["Cut size = ", n3];
Print["Fractal dimension = ", df];
Print["Fractal codimension = ", (2 - df)];
{akaike, bayes, determination} = line[{"AIC", "BIC", "RSquared"}];
Print["Akaike criteria = ", akaike]
Print["Bayes criteria = ", bayes]
Print["Determination coefficient = ", determination]

```

COMMENT A.6.— This program calculates the variogram dimension for a set of data. The value of this dimension is not as precise when values away from the input of an irregular variogram are used. The user may therefore choose a limit from which he or she no longer takes account of this data. The autovariance function was inspired by W. C. Haneberg's book [HAN 04].

PROGRAM A.7.—

```

(*Entropy fractal dimension of a set of data*)
(*The series studied should have 32, 64, 128, 256, 512, 1024,... values*)
(*Series import*)
ClearAll["Global`*"]
d = Import[SystemDialogInput["FileOpen"]];
ima = Flatten[d];
n1 = Length[d];
(*Calculation of entropy values for each subset*)

```

```
f[n_] := Module[{xx = ima, d2, tot, freq, x, y, logProduct, ent, dx, dxm},
  d2 = Partition[xx, n];
  tot = Map[(Apply[Plus, #]) &, d2];
  freq = N[d2/tot];
  logProduct[x_, y_] := x Log[2, y] /; x != 0 || y != 0;
  logProduct[x_, y_] := 0.0 /; x == 0 && y == 0;
  ent[list_] := -Plus @@ logProduct[list, list];
  SetAttributes[logProduct, Listable];
  dx = Map[ent, freq] // N;
  dxm = Mean[dx]
]
d3 = Table[f[i], {i, {4, 8, 16, 32, 64, 128, 256}}]
logtaille = Log[{4, 8, 16, 32, 64, 128, 256}] // N
d4 = Riffle[logtaille, d3];
d6 = Partition[d4, 2];
Print["Spectrum and adjustment values"]
line = LinearModelFit[d6, {1, x}, x];
linen = Normal[line];
da = linen[[2, 1]];
Print["Fractal dimension = ", da];
Print["Fractal codimension = ", 2 - da];
{akaike, bayes, determination} = line[{"AIC", "BIC", "RSquared"}];
Print["Akaike criteria = ", akaike]
Print["Bayes criteria = ", bayes]
Print["Determination coefficient = ", determination]
ListPlot[{d6}, ImageSize -> {300, 200}, Joined -> True,
  AxesLabel -> {"Log_size", "Entropy"}, PlotRange -> All,
  PlotStyle -> {Black}]
```

COMMENT A.7.- This program calculates the entropy dimension. It could be simplified using the Entropy[] instruction.

PROGRAM A.8.-

```
(*Calculation of the fractal dimension using the power spectrum*)
(*Import and visualization of a set of data containing 64, 128, 256, ... records*)
ClearAll["Global`*"]
d = Import[SystemDialogInput["FileOpen"]];
d = Flatten[d];
Print["origin series"]
Print["series length = ", n1 = Length[d]]
ListPlot[{d}, ImageSize -> {500, 400}, Filling -> Axis, Joined -> True,
```

```

AxesLabel -> {"time", "Magnitude"}, PlotRange -> All,
PlotStyle -> PointSize[Medium]]
(*Calculation of the Fourier spectrum*)
d = d - Mean[d];
n2 = n1/2;
n3 = n1 + 1;
Print[]
n8 = DialogInput[
DynamicModule[{name = ""},
Column[{
"Number of spectrum records to be excluded from the adjustment",
TextField[Dynamic[name], String],
ChoiceButtons[{DialogReturn[name], DialogReturn[]}]
}]]];
n8 = ToExpression[n8];
Print["Number of spectrum records to be excluded from the adjustment = ", n8]
wi = Table[1 - (((2*i)/n3) - 1)^2, {i, 1, n1}] // N;
d = d*wi;
spectr = Abs[Chop[Fourier[d]]];
spectrum = N[spectr^2];
spectre = Log[Take[spectrum, {n8, n2}]];
(*Bilogarithmic regression adjustment*)
d4 = Table[i, {i, n8, n2}];
d4 = Log[1/d4];
d4min = Min[d4];
d4max = Max[d4];
dspecmin = Min[spectre];
d5 = Riffle[d4, spectre];
d6 = Partition[d5, 2];
Print[]
Print["Spectrum values and adjustment"]
line = LinearModelFit[d6, {1, x}, x];
linen = Normal[line]
Show[ListPlot[d6], Plot[line[x], {x, d4min, d4max}], Frame -> True]
Print[]
da = linen[[2, 1]];
Print["Process type"]
Which[da < -3, "Attention biais", -3 < da < -1, "DfGn", -1 < da < 1, "fGn",
da == 0, "Bm", +1 < da < +3, "fBm", +3 < da < +5, "fBml", +5 <
da, "Attention biais"]
Print["Hurst coefficient = "]
Which[da < -3, "Attention biais", -3 < da < -1, h = (da + 3)/2, -1 < da < 1,
h = (da + 1)/2, da == 0, h = 0.5, 1 < da < 3, h = (da - 1)/2, 3 < da < 5,

```

```

h = (da - 3)/2, 5 < da, "Attention biais"]
Print["Fractal dimension = ", 2 - h];
Print["Fractal codimension = ", 2 - (2 - h)];
{akaike, bayes, determination} = line[{ "AIC", "BIC", "RSquared"}];
Print["Akaike criteria = ", akaike]
Print["Bayes criteria = ", bayes]
Print["Determination coefficient = ", determination]

```

COMMENT A.8.— This program calculates the dimension of the energy spectrum determined following a Fourier decomposition. As rapid decomposition is used, the number of records in the series should be adapted. Various corrections are included in order to obtain a regular spectrum.

PROGRAM A.9.—

```

(*Wavelet decomposition of a unique series and calculation of the fractal
dimension*)
(*Data import*)
ClearAll["Global`*"]
d = Import[SystemDialogInput["FileOpen"]];
d = Flatten[d];
d = d - Mean[d];
Print["series length = ", n1 = Length[d]]
ListPlot[{d}, ImageSize -> {500, 400}, Filling -> Axis, Joined -> True,
  AxesLabel -> {"time", "Magnitude"}, PlotRange -> All,
  PlotStyle -> PointSize[Medium]]
(*Choose a filter order and decomposition level*)
dwd1 = DiscreteWaveletTransform[d, DaubechiesWavelet[6], 5];
WaveletListPlot[dwd1, PlotLayout -> "CommonXAxis", Filling -> Axis]
WaveletScalogram[dwd1, {___, 1}, Method -> "Inverse" -> True,
  ColorFunction -> "BlueGreenYellow"]
(*Calculation of mean spectrum and adjustment*)
spectre =
  Mean[Abs[Flatten[#]]] & /@ dwd1[{___, 1}, "Values"]
n1 = Length[spectre]
d4 = Table[i, {i, 1, n1}];
d4 = Log[d4];
d4min = Min[d4];
d4max = Max[d4];
spectre = Log[spectre];
d5 = Riffle[d4, spectre];
d6 = Partition[d5, 2];

```

```

Print[]
Print["Spectrum value and adjustment"]
line = LinearModelFit[d6, x, x];
linen = Normal[line]
Show[ListPlot[d6], Plot[line[x], {x, d4min, d4max}], Frame -> True]
Print[]
da = linen[[2, 1]];
hur = da - 0.5
Print["Process type"]
Which[da < -3, "Attention biais", -3 < da < -1, "DfGn", -1 < da < 1, "fGn",
da == 0, "Bm", +1 < da < +3, "fBm", +3 < da < +5, "fBmI", +5 <
da, "Attention biais"]
Print["Hurst coefficient = "]
Which[da < -3, "Attention biais", -3 < da < -1, h = (da + 3)/2, -1 < da < 1,
h = (da + 1)/2, da == 0, h = 0.5, 1 < da < 3, h = (da - 1)/2, 3 < da < 5,
h = (da - 3)/2, 5 < da, "Attention biais"]
Print["Fractal dimension = ", 2 - h];
Print["Fractal codimension = ", 2 - (2 - h)];
{akaike, bayes, determination} = line[{ "AIC", "BIC", "RSquared"}];
Print["Akaike criteria = ", akaike]
Print["Bayes criteria = ", bayes]
Print["Determination coefficient = ", determination]

```

COMMENT A.9.— This program calculates the dimension of the spectrum determined following a wavelet decomposition. It only works with version 8 of *Mathematica*. It also gives figures for the wavelet decomposition, which are very useful in any interpretation.

#### A.5. Software programs for calculating the fractal dimensions of a satellite image or map

We are proposing three programs for image processing. The first calculates the fractal dimensions of the variogram in west–east and north–south directions. The other two determine fractal dimensions from the energy spectrums from Fourier and wavelet decompositions.

PROGRAM A.10.—

```

(*Calculation of autocorrelations, variograms and fractal dimensions of an image*)
(*Image import*)
ClearAll["Global`*"]
donimage = Import[SystemDialogInput["FileOpen"]];

```

```
Show[donimage]
d = N[ImageData[donimage, "Byte"]];
(*Calculation of the two row and column means and their variances*)
d1 = Mean[d];
d2 = Mean[Transpose[d]];
n1 = Length[d] ;
n2 = n1/2;
v1 = Variance[d1];
vc = Variance[d2];
(*Calculation of row and column autocorrelations*)
Autocorrelation[x_, dt_] := Return[Correlation[x, RotateLeft[x, dt]]]
d3 = Table[Autocorrelation[d1, n1], {n1, 0, n1/2}];
d4 = Table[Autocorrelation[d2, n1], {n1, 0, n1/2}];
ListPlot[{d3, d4}, ImageSize -> {500, 400}, AxesOrigin -> {0, 0},
  Filling -> Axis, AxesLabel -> {"dt", "Autocorrelation"},
  PlotStyle -> PointSize[Medium]]
Print[]
Print["The West-East autocorrelation row is in Blue, and the North-South
autocorrelation column is in Pink"]
Print[]
Clear[d3, d4];
Autocovariance[y_, dtemp_] := Return[Covariance[y, RotateLeft[y, dtemp]]]
d3 = Table[v1 - Autocovariance[d1, n1], {n1, 0, n1/2}];
d4 = Table[vc - Autocovariance[d2, n1], {n1, 0, n1/2}];
ListPlot[{d3, d4}, ImageSize -> {500, 400}, Filling -> Axis,
  AxesLabel -> {"dt", "Variogram"}, PlotStyle -> PointSize[Medium]]
Print[]
Print["The West-East variogram row is in Blue, and the North-South variogram
column is in Pink"]
Print[]
n3 = DialogInput[
  DynamicModule[{name = ""}, Column[{
    "Choose the cut size",
    InputField[Dynamic[name], String],
    ChoiceButtons[{DialogReturn[name], DialogReturn[]}]
  }]]];
n3 = ToExpression[n3];
d5 = Table[i, {i, 1, n2 - 1, 1}];
d3 = Drop[d3, 1];
d5a = Riffle[d5, d3];
d6 = Partition[d5a, 2];
d6 = Take[d6, n3];
d6 = Log[d6];
```

```

Print["West-East direction result = "]
line = LinearModelFit[d6, {1, x}, x];
linen = Normal[line];
da = linen[[2, 1]];
df = N[(4 - Abs[da])/2];
hurst = N[2 - df];
Print["Fractal dimension = ", df];
{akaike, bayes, determination} = line[{ "AIC", "BIC", "RSquared"}];
Print["Akaike criteria = ", akaike]
Print["Bayes criteria = ", bayes]
Print["Determination coefficient = ", determination]
Print["East-West Hurst coefficient = ", hurst]
Print[]
(*Calculation of North-South log and fractal dimension*)
Clear[line, linen, da, df, hurst]
d5 = Table[i, {i, 1, n2 - 1, 1}];
d4 = Drop[d5, 1];
d5a = Riffle[d5, d4];
d6 = Partition[d5a, 2];
d6 = Take[d6, n3];
d6 = Log[d6];
Print["North-South result = "]
line = LinearModelFit[d6, {1, x}, x];
linen = Normal[line];
da = linen[[2, 1]];
df = N[(4 - Abs[da])/2];
hurst = N[2 - df];
Print["Fractal dimension = ", df];
{akaike, bayes, determination} = line[{ "AIC", "BIC", "RSquared"}];
Print["Akaike criteria = ", akaike]
Print["Bayes criteria = ", bayes]
Print["Determination coefficient = ", determination]
Print["North-South Hurst coefficient = ", hurst]

```

COMMENT A.10.— This program is not optimized so as to be readable by a novice. It would be easy to reduce it by using the module function and by stacking multiple instructions.

PROGRAM A.11.—

```

(*Calculation of the fractal dimension using the power spectrum*)
(*Data import*)

```

```

ClearAll["Global`*"]
donimage = Import[SystemDialogInput["FileOpen"]];
Show[donimage]
d = N[ImageData[donimage, "Byte"]];
n1 = Length[d];
n2 = n1/4;
(*Calculation over the whole image*)
Print[]
Print["Image size = ", n1]
(*Calculation of the Fourier spectrum*)
spectr = Chop[Fourier[d]];
spectrum = Abs[spectr];
spectrum = N[spectrum^2/Sqrt[n1]];
spectri = Take[spectrum, {n2, 3*n2}, {n2, 3*n2}];
ListDensityPlot[spectri]
spectrum = Flatten[spectrum];
spectre = Take[spectrum, {2, n2}];
(*Bilogarithmic regression adjustment*)
d4 = Table[i, {i, 1, n2 - 1}];
d4 = N[1/d4];
d4 = Log[d4];
d4min = Min[d4]
d4max = Max[d4]
spectre = Log[spectre];
dspecmin = Min[spectre]
dspecmax = Max[spectre]
d5 = Riffle[spectre, d4];
d6 = Partition[d5, 2];
Print[]
Print["Spectrum values and adjustment"]
line = LinearModelFit[d6, {1, x}, x];
linen = Normal[line]
Show[ListPlot[d6], Plot[line[x], {x, d4max, dspecmax}], Frame -> True]
Print[]
da = linen[[2, 1]]
Print["Process type"]
Which[da < -3, "Attention biais", -3 < da < -1, "DfGn", -1 < da < 1, "fGn",
da == 0, "Bm", +1 < da < +3, "fBm", +3 < da < +5, "fBmI", +5 <
da, "Attention biais"]
Print["Hurst coefficient = "]
Which[da < -3, "Attention biais", -3 < da < -1, h = (da + 3)/2, -1 < da < 1,

```

```

h = (da + 1)/2, da == 0, h = 0.5, 1 < da < 3, h = (da - 1)/2, 3 < da < 5,
h = (da - 3)/2, 5 < da, "Attention biais"]
df = 3 - h;
Print["Fractal dimension = ", df];
Print["Fractal codimension = ", 3 - df];
{akaike, bayes, determination} = line[{"AIC", "BIC", "RSquared"}];
Print["Akaike criteria = ", akaike]
Print["Bayes criteria = ", bayes]
Print["Determination coefficient = ", determination]

```

COMMENT A.11.— The image handled should be a minimum of 256 pixels squared.

PROGRAM A.12.—

```

(*Calculation of the fractal dimension of an image using the energy spectrum
following a wavelet decomposition*)
(*Data import*)
ClearAll["Global`*"]
donimage = Import[SystemDialogInput["FileOpen"]];
Show[donimage]
d = N[ImageData[donimage, "Byte"]];
n1 = Length[d];
n2 = n1/4;
dwd1 = DiscreteWaveletTransform[donimage, DaubechiesWavelet[6], 5]
func[x_, {___, 1 | 2 | 3}] :=
  ImageAdjust[ImageAdjust[ImageApply[Abs, x]], {0, 2}];
func[x_, wind_] := ImageAdjust[x];
WaveletImagePlot[dwd1, Automatic, func]
Table[Labeled[
  WaveletImagePlot[dwd1, Automatic, Composition[f, ImageAdjust][#] &],
  f], {f, {Blur, Sharpen, Dilation[#, 2] &}}]
energy[data_] := Sort[Flatten[data]^2, Greater]
spectre = energy[Last /@ dwd1[Automatic]];
n2 = Length[spectre]/2;
d4 = Table[i, {i, 1, n2}];
d4 = Log[d4];
d4max = Max[d4];
spectre = Take[spectre, {1, n2}];
spectre = Log[spectre];
dspecmin = Min[spectre];
d5 = Riffle[spectre, d4];

```

```

d6 = Partition[d5, 2];
Print[]
Print["Spectrum values and adjustment"]
line = LinearModelFit[d6, x, x];
linen = Normal[line];
Show[ListPlot[d6], Plot[line[x], {x, dspecmin, d4max}], Frame -> True]
Print[]
da = linen[[2, 1]]
Print["Process type"]
Which[da < -3, "Attention biais", -3 < da < -1, "DfGn", -1 < da < 1, "fGn",
da == 0, "Bm", +1 < da < +3, "fBm", +3 < da < +5, "fBmI", +5 <
da, "Attention biais"]
Print["Hurst coefficient = "]
Which[da < -3, "Attention biais", -3 < da < -1, h = (da + 3)/2, -1 < da < 1,
h = (da + 1)/2, da == 0, h = 0.5, 1 < da < 3, h = (da - 1)/2, 3 < da < 5,
h = (da - 3)/2, 5 < da, "Attention biais"]
df = 3 - h;
Print["Fractal dimension = ", df];
Print["Fractal codimension = ", 3 - df];
{akaike, bayes, determination} = line[{ "AIC", "BIC", "RSquared"}];
Print["Akaike criteria = ", akaike]
Print["Bayes criteria = ", bayes]
Print["Determination coefficient = ", determination]

```

COMMENT A.12.— The program outputs several graphs for interpretation of the wavelet decomposition. Other wavelet choices could be introduced.

#### A.6. Software programs for calculating multifractal spectrums of a series and an image

We are proposing four programs for the multifractal approach. The first and second are simple visual tests on a set of data or an image. The other two calculate multifractal spectrums for a series and an image.

PROGRAM A.13.—

```

(*Testing the multifractality of a series*)
(*Data import*)
ClearAll["Global`*"]
d = Import[SystemDialogInput["FileOpen"]];
d = Flatten[d];

```

```

n1 = Length[d];
ListPlot[{d}, ImageSize -> {500, 400}, PlotRange -> All, Filling -> Axis,
Joined -> True, AxesLabel -> {"time", "Magnitude"},
PlotStyle -> PointSize[Medium]]
Table[Histogram[d, b,
PlotLabel -> b], {b, {"Sturges", "Scott", "FreedmanDiaconis", "Wand"}}]
d1 = RotateLeft[d, 1];
d2 = Abs[d1 - d];
Print["histogram of differentiated data"]
Print["The mode may be small and the decrease slow"]
Table[Histogram[d2, b,
PlotLabel -> b], {b, {"Sturges", "Scott", "FreedmanDiaconis", "Wand"}}]

```

COMMENT A.13.— The test is performed over four histogram categories, which increases its validity. The only outputs from the program are graphs.

PROGRAM A.14.—

```

(*Visual test of image multifractality*)
(*Image data imported at the im instruction*)
ClearAll["Global`*"]
im = Import[SystemDialogInput["FileOpen"]];
Show[im]
prewitY = {{-1, -1, -1}, {0, 0, 0}, {1, 1, 1}}/3.;
prewitX = {{1, 0, -1}, {1, 0, -1}, {1, 0, -1}}/3.;
imgrad = Sqrt[
ImageData[ImageConvolve[im, prewitX]]^2 +
ImageData[ImageConvolve[im, prewitY]]^2];
imgrad1 = Flatten[imgrad];
Table[Histogram[imgrad1, b,
PlotLabel -> b], {b, {"Sturges", "Scott", "FreedmanDiaconis", "Wand"}}]

```

COMMENT A.14.— The calculation is performed following a double convolution in order to determine the gradient.

PROGRAM A.15.—

```

(*Multifractal spectrum over a series*)
(*Series import*)
ClearAll["Global`*"]
d = Import[SystemDialogInput["FileOpen"]];
ima = Flatten[d];
n1 = Length[d];

```

```
(*Calculation of Dq values*)
f[n_] := Module[{xx = ima, d2, stot, freqpuis, somfreqpuis, pq},
  d2 = Partition[xx, n];
  fmaxmin[x_List] := Mean[x];
  stot = Map[fmaxmin, d2] // N;
  stot = stot/Total[Flatten[stot]];
  freqpuis = Table[stot^j, {j, -5, 5, 0.5}];
  somfreqpuis = Total[Transpose[freqpuis]];
  pq = Log[2, somfreqpuis]
]
logmuql = Flatten[Table[f[i], {i, {4, 8, 16, 32, 64, 128, 256}}]];
logmuqlt = Partition[logmuql, 21];
pqpart = logmuqlt;
logtaille = Log[2, {4, 8, 16, 32, 64, 128, 256}] // N;
f2[nn_] := Module[{a = logtaille, b = logmuqlt, essai, line, linen, fa1},
  essai = Partition[Riffle[logtaille, logmuqlt[[All, nn]]], 2];
  line = LinearModelFit[essai, x, x];
  linen = Normal[line];
  fa1 = linen[[2, 1]] // N
]
Dq = Table[f2[i], {i, 1, 21, 2}];
qq = {-5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5};
q = 1/(qq - 1);
Dqq = q*Dq;
kq = 2 - Dqq;
cq = kq*q;
qmoin = qq - 1;
rq = qmoin*Dq;
ecartDq = Dqq[[1]] - Dqq[[11]] // N;
Print["Dq deviation = ", ecartDq]
ListPlot[{pqpart[[All, 1]], pqpart[[All, 5]], pqpart[[All, 10]],
  pqpart[[All, 15]], pqpart[[All, 21]]}, ImageSize -> {300, 200},
  Joined -> True, AxesLabel -> {"log_BoxArea", "SumPiQ"}, PlotRange -> All,
  PlotStyle -> PointSize[Medium]]
(*Calculation of values for other spectrums and graph outlines *)
dqrel = Dqq - 1;
d3 = Riffle[qq, Dqq];
d6 = Partition[d3, 2];
d6 = Drop[d6, {7}];
ListPlot[{d6}, ImageSize -> {300, 200}, Joined -> True,
  AxesLabel -> {"q", "Dq"}, PlotRange -> All, PlotStyle -> {Black}]
```

```

d3 = Riffle[qq, kq];
d6 = Partition[d3, 2];
d6 = Drop[d6, {7}];
ListPlot[{d6}, ImageSize -> {300, 200}, Joined -> True,
  AxesLabel -> {"q", "Kq"}, PlotRange -> All, PlotStyle -> {Black}]
d3 = Riffle[qq, cq];
d6 = Partition[d3, 2];
d6 = Drop[d6, {7}];
ListPlot[{d6}, ImageSize -> {300, 200}, Joined -> True,
  AxesLabel -> {"q", "Cq"}, PlotRange -> All, PlotStyle -> {Black}]
d3 = Riffle[qmoin, rq];
d6 = Partition[d3, 2];
d6 = Drop[d6, {7}];
ListPlot[{d6}, ImageSize -> {300, 200}, Joined -> True,
  AxesLabel -> {"q", "Tauq"}, PlotRange -> All, PlotStyle -> {Black}]
d3 = Riffle[qq, dqrel];
d6 = Partition[d3, 2];
d6 = Drop[d6, {7}];
ListPlot[{d6}, ImageSize -> {300, 200}, Joined -> True,
  AxesLabel -> {"q", "Relative_Dq"}, PlotRange -> All, PlotStyle -> {Black}]
fo[n_] :=
Module[{xx = ima, d2, stot, tfreqpuis, freqpuis, logfreqpuis, produitfreq,
  somfproduit4},
  d2 = Partition[xx, n];
  fmaxmin[x_List] := Mean[x];
  stot = Map[fmaxmin, d2] // N;
  freqpuis = Table[stot^j, {j, -5, 5, 0.5}];
  tfreqpuis = Total[Transpose[freqpuis]];
  freqpuis = freqpuis/tfreqpuis;
  logfreqpuis = Log[10, freqpuis];
  produitfreq = freqpuis*logfreqpuis;
  somfproduit4 = Map[Total, produitfreq] // N
]
totlogmuql = Flatten[Table[fo[i], {i, {4, 8, 16, 32, 64, 128, 256}}]];
logmuqlt = Partition[totlogmuql, 21];
pqpart = logmuqlt;
logtaille = Log[10, {4, 8, 16, 32, 64, 128, 256}] // N;
fo2[nn_] := Module[{a = logtaille, b = logmuqlt, essai, line, linen, fa1},
  essai = Partition[Riffle[logtaille, logmuqlt[[All, nn]]], 2];
  line = LinearModelFit[essai, x, x];
  linen = Normal[line];

```

```

fa1 = ToExpression[linen[[2, 1]]] // N
]
fqr = Table[fo2[i], {i, 1, 21, 2}];
qq = {-5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5};
fq2 = Riffle[qq, fqr];
fq2 = Partition[fq2, 2];
ListPlot[{fq2}, ImageSize -> {300, 200}, Joined -> True,
  AxesLabel -> {"q", "f(alpha)q"}, PlotRange -> All, PlotStyle -> {Black}]
ecartFq = fqr[[1]] - fqr[[11]] // N;
Print["Fq deviation = ", ecartFq]

```

COMMENT A.15.— This program is for academic use. It is not optimized. The spectrums are calculated over a reduced interval.

PROGRAM A.16.—

```

(*Calculation of multifractal spectrums for an image*)
(*Image import*)
ClearAll["Global`*"]
ima = Import[SystemDialogInput["FileOpen"]];
Show[ima]
d = N[ImageData[ima, "Byte"]];
n1 = Length[d];
(*Calculation of Dq values*)
f[n_] := Module[{xx = ima, imagepart4, d2, sstot, stot, freqpuis, somfreqpuis,
  pq4},
  imagepart4 = Flatten[ImagePartition[xx, n]];
  d2 = Map[ImageData[#, "Byte"] &, imagepart4] // N;
  d2 = d2/Total[Flatten[d]];
  sstot = Total[Transpose[d2]];
  stot = Map[Total, sstot];
  freqpuis = Table[stot^j, {j, -5, 5, 0.5}];
  somfreqpuis = Total[Transpose[freqpuis]];
  pq4 = Log[10, somfreqpuis]
]
logmuql = Flatten[Table[f[i], {i, {4, 8, 16, 32, 64, 128, 256}}]];
logmuqlt = Partition[logmuql, 21];
pqpart = logmuqlt;
logtaille = Log[10, {4, 8, 16, 32, 64, 128, 256}] // N;
f2[nn_] := Module[{a = logtaille, b = logmuqlt, essai, line, linen, fa1},
  essai = Partition[Riffle[a, b[[All, nn]]], 2];
  line = LinearModelFit[essai, x, x];

```

```

    linen = Normal[line];
    fa1 = linen[[2, 1]] // N
  ]
  Dq = Table[f2[i], {i, 1, 21, 2}];
  qq = {-5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5};
  q = 1/(qq - 1);
  Dqq = q*Dq;
  kq = 2 - Dqq;
  cq = kq*q;
  qmoin = qq - 1;
  rq = qmoin*Dq;
  ecartDq = Dqq[[1]] - Dqq[[11]] // N;
  Print["Dq deviation = ", ecartDq]
  ListPlot[{pqpart[[All, 1]], pqpart[[All, 5]], pqpart[[All, 10]],
    pqpart[[All, 15]], pqpart[[All, 21]]}, ImageSize -> {300, 200},
    Joined -> True, AxesLabel -> {"log_BoxArea", "SumPiQ"}, PlotRange -> All,
    PlotStyle -> PointSize[Medium]]
  (*Calculation of values of other spectrums*)
  dqrel = Dqq - 1;
  d3 = Riffle[qq, Dqq];
  d6 = Partition[d3, 2];
  d6 = Drop[d6, {7}];
  ListPlot[{d6}, ImageSize -> {300, 200}, Joined -> True,
    AxesLabel -> {"q", "Dq"}, PlotRange -> All, Frame -> True,
    PlotStyle -> {Black}]
  d3 = Riffle[qq, kq];
  d6 = Partition[d3, 2];
  d6 = Drop[d6, {7}];
  ListPlot[{d6}, ImageSize -> {300, 200}, Joined -> True,
    AxesLabel -> {"q", "Kq"}, PlotRange -> All, Frame -> True,
    PlotStyle -> {Black}]
  d3 = Riffle[qq, cq];
  d6 = Partition[d3, 2];
  d6 = Drop[d6, {7}];
  ListPlot[{d6}, ImageSize -> {300, 200}, Joined -> True,
    AxesLabel -> {"q", "Cq"}, PlotRange -> All, Frame -> True,
    PlotStyle -> {Black}]
  d3 = Riffle[qmoin, rq];
  d6 = Partition[d3, 2];
  d6 = Drop[d6, {7}];
  ListPlot[{d6}, ImageSize -> {300, 200}, Joined -> True,

```

```

AxesLabel -> {"q", "Tauq"}, PlotRange -> All, Frame -> True,
PlotStyle -> {Black}]
d3 = Riffle[qq, dqrel];
d6 = Partition[d3, 2];
d6 = Drop[d6, {7}];
ListPlot[{d6}, ImageSize -> {300, 200}, Joined -> True,
AxesLabel -> {"q", "Relative_Dq"}, PlotRange -> All, Frame -> True,
PlotStyle -> {Black}]
fa[n_] :=
Module[{xx = ima, imagepart4, d2, fmaxmin, stot, freqpuis, tfreqpuis,
somfreqpuis, logfreqpuis, produitfreq, somfproduit4},
imagepart4 = Flatten[ImagePartition[xx, n]];
d2 = Map[ImageData[#, "Byte"] &, imagepart4] // N;
fmaxmin[x_List] := Max[x] - Min[x];
stot = Map[fmaxmin, d2] // N;
freqpuis = Table[stot^j, {j, -5, 5, 0.5}];
tfreqpuis = Total[Transpose[freqpuis]];
freqpuis = freqpuis/tfreqpuis;
logfreqpuis = Log[10, freqpuis];
produitfreq = freqpuis*logfreqpuis;
somfproduit4 = Map[Total, produitfreq] // N
]
logmuql = Flatten[Table[fa[i], {i, {4, 8, 16, 32, 64, 128, 256}}]];
logmuqlt = Partition[logmuql, 21];
pqpart = logmuqlt;
logtaille = Log[10, {4, 8, 16, 32, 64, 128, 256}] // N;
fb2[nn_] := Module[{a = logtaille, b = logmuqlt, essai, line, linen, fa1},
essai = Partition[Riffle[a, b[[All, nn]]], 2];
line = LinearModelFit[essai, x, x];
linen = Normal[line];
fa1 = linen[[2, 1]] // N
]
fqr = Table[fb2[i], {i, 1, 21, 2}];
qq = {-5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5};
fq2 = Riffle[qq, fqr];
fq2 = Partition[fq2, 2];
maxfq = Max[fqr];
minfq = Min[fqr];
Print["fq deviation = ", maxfq - minfq]
ListPlot[{pqpart[[All, 1]], pqpart[[All, 5]], pqpart[[All, 10]],
pqpart[[All, 15]], pqpart[[All, 21]]}, ImageSize -> {300, 200},

```

```
Joined -> True, AxesLabel -> {"log_BoxArea", "SumPiQ"}, PlotRange -> All,
PlotStyle -> PointSize[Medium]]
ListPlot[fq2, ImageSize -> {300, 200}, Joined -> True,
AxesLabel -> {"q", "f(alpha)q"}, PlotRange -> All, PlotStyle -> {Black}]
```

COMMENT A.16.— The same remarks apply here as for the previous program.

## Bibliography

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## Index

### A, B, C

abnormal diffusion, 37, 55, 146  
Akaïkè criteria, 97  
auto-affine fractal, 80  
auto-similar, 2  
    fractal, 22, 25-26, 62, 75, 79, 85, 185  
    auto-similarity, 10, 18, 22, 76, 123, 125, 179  
Bayes criteria, 199-201, 204-213  
box counting dimension, 77  
Brownian diffusion, 37, 188  
    motion, 25-28, 31-35, 129, 141, 156  
center/periphery model, 130, 187  
chaos theory, 170  
Clarck model, 131-132  
codimension, 39-42, 56, 117  
complexity theory, 138-140, 164, 187, 188  
constructal theory, 144, 151, 152  
critical self-organization theory, 5, 144, 147, 150, 151

### D, E

differentiated fractional Gaussian noise, 72-73, 80  
diffusion entropy analysis, 63-64  
dominant coefficient method, 112  
endogenous development, 150, 187  
exogenous development, 187  
exponential distribution, 91-93, 99, 168, 194

### F

Fourier decomposition, 25, 63-69, 72-74, 81, 84-85, 193, 198, 207, 208  
fractal  
    dimension, 1-2, 5-7, 10, 15-21, 24-28, 31, 34-45, 48-53, 56, 59, 62-64, 72-84, 88, 91, 95-98, 103, 110-112, 128-131, 147, 164, 173-185, 191, 198, 203, 205-207, 210-212  
    networks, 171  
    parabolic, 130, 171  
    paradigm, 3, 5-6, 128, 178-184, 188  
    perturbation, 123-125, 186

scanning, 40, 48-50, 57, 110, 184, 191  
simulation, 5-6, 172  
fractional  
Brownian motion, 29-32, 35, 64, 80-81  
fractional Gaussian noise, 29, 31, 64-66, 73-74, 81, 84

## G, H, I

general theory of morphogenesis, 154  
geostatistical, 5, 84  
geostatistics, 6, 173, 184, 191  
Gibrat distribution, 134  
law, 46, 96, 134  
Gutenberg-Richter  
distribution, 126  
law, 5  
Hausdorff-Besicovitch dimension, 10  
hierarchical diffusion, 29, 33, 135, 161  
Hurst coefficient, 27-32, 53, 63-64, 68, 72-74, 80-81, 84, 179  
image  
analysis, 48  
processing, 6, 77, 167, 173-176, 184, 208  
information dimension, 56, 77, 175  
integrated fractional Brownian  
motion, 72, 80  
intermittence, 105, 151, 162, 170, 188

## L, M

lacunarity, 39-42, 77-79, 175  
least action principle, 151, 159  
Lévy  
model, 32-34, 62-64  
motion, 33-34  
movement, 179

log-normal  
distribution, 14-16, 45-46, 92-93, 96-98, 101, 133-134  
law, 97  
log-periodic distribution, 47, 126, 154, 170  
multi-agent system, 7, 36, 141, 158, 161  
multifractal, 3, 6-9, 19, 49-56, 60, 103-112, 115-118, 125, 128, 131-132, 142, 174-176, 184-185, 193  
scanning, 117  
multi-scale, 4, 15-19, 52, 56, 67-71, 79, 94, 111-112, 117-119, 130, 137, 177, 182-189

## N, P

normal  
diffusion, 26-28, 139, 158-160, 184  
distribution, 14, 17-19, 26-28, 91  
Norwegian  
perturbance, 120  
perturbation, 120-125  
parabolic fractal, 43-46, 97-99, 101  
Pareto  
distribution, 12, 15-18, 43, 101, 130, 133, 137, 169  
law, 90-91, 96-98  
power law, 1-4, 8, 9-18, 26, 32-34, 39, 43-46, 89-99, 126-127, 131, 132, 135-136, 147, 168-169, 194, 202  
principle of least action, 133

## R, S

rank-size  
distribution, 44, 96-99, 100-101, 104-106, 133-135, 143, 146-150, 159, 162, 171, 196  
rule, 1, 8, 43  
risk, 19, 46, 91, 126-128, 147, 167-172, 187

scale  
   invariance, 8-10, 18, 24, 46, 134,  
   137, 143, 152, 185-187  
   relativity theory, 144, 153-154  
 scaling parameter, 2-4, 15-16, 19,  
   65-67, 81, 88-90, 96, 135, 169, 182  
 self-affine fractal, 24-25, 62-63,  
   84-85  
 semi-fractal, 8, 43-46, 94, 137,  
   182-183  
 spectral fractal dimension, 80

## **T, U, V, W, Z**

topological dimension, 2, 10, 15, 28,  
   34, 41-42, 73, 78-80  
 turbulence, 36, 50, 57, 64, 105, 112,  
   117, 120-126, 129, 137, 144, 157,  
   161, 162-163, 184-186

under-determination, 141  
 urban hierarchy, 18, 46, 98, 106-108,  
   114, 132-134, 159, 183  
 variogram dimension, 204  
 wavelet decomposition, 25, 63-64,  
   67-75, 81, 84-85, 112, 173-176,  
   198, 208, 212-213  
 Zipf  
   distribution, 133-134, 199  
   law, 9, 16, 90-91, 95-98, 132, 144